Chapter 3

Linear Transformations and Matrix Algebra

Where are we?

In Chapter 1 we learned to solve all linear systems algebraically.

In Chapter 2 we learned to think about the solutions geometrically.

In Chapter 3 we continue with the algebraic abstraction. We learn to think about solving linear systems in terms of inputs and outputs. This is similar to control systems in AE, objects in computer programming, or hot pockets in a microwave.

More specifically, we think of a matrix as giving rise to a function with inputs and outputs. Solving a linear system means finding an input that produces a desired output. We will see that sometimes these functions are invertible, which means that you can reverse the function, inputting the outputs and outputting the inputs.

The invertible matrix theorem is the highlight of the chapter; it tells us when we can reverse the function. As we will see, it ties together everything in the course so far.

Sections 3.1

Matrix Transformations

Section 3.1 Outline

- Learn to think of matrices as functions, called matrix transformations
- Learn the associated terminology: domain, codomain, range
- Understand what certain matrices do to $\mathbb{R}^n$

From matrices to functions

Let A be an $m \times n$ matrix.

We define a function

$$T : \mathbb{R}^n \to \mathbb{R}^m$$
$$T(v) = Av$$

This is called a matrix transformation.

The domain of T is $\mathbb{R}^n$.

The co-domain of T is $\mathbb{R}^m$.

The range of T is the set of outputs: $\mathrm{Col}(A)$

This gives us a*nother* point of view of $Ax = b$

Demo

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$.

What is $T(u)$?

Find v in $\mathbb{R}^2$ so that $T(v) = b$

Find a vector in $\mathbb{R}^3$ that is not in the range of T.

Square matrices

For a square matrix we can think of the associated matrix transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

as doing something to $\mathbb{R}^n$.

Example. The matrix transformation T for

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

What does T do to $\mathbb{R}^2$?

Square matrices

What does each matrix do to $\mathbb{R}^2$?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

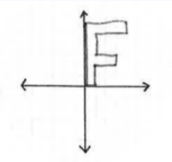
$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

What is the range in each case?

Poll

Poll

What does $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ do to this letter F?



Square matrices

What does each matrix do to $\mathbb{R}^2$?

Hint: if you can't see it all at once, see what happens to the x- and y-axes.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Examples in $\mathbb{R}^3$

What does each matrix do to $\mathbb{R}^3$?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Why are we learning about matrix transformations?

Sample applications:

- Cryptography (Hill cypher)
- Computer graphics (Perspective projection is a linear map!)
- Aerospace (Control systems - input/output)
- Biology
- Many more!

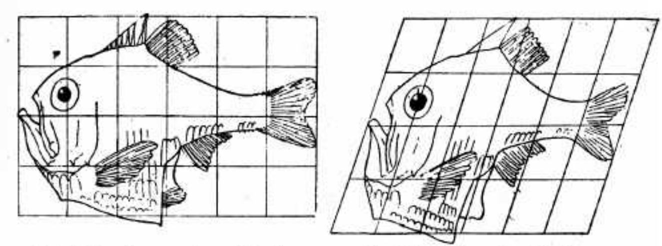


Fig. 517. *Argyropelecus Olfersi.* Fig. 518. *Sternoptyx diaphana.*

Applications of Linear Algebra

Biology: In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population in 2016 (in terms of the number of first, second, and third year rabbits), then what is the population in 2017?

These relations can be represented using a matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

How does this relate to matrix transformations?

Demo

Section 3.1 Summary

- If A is an $m \times n$ matrix, then the associated matrix transformation T is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\mathrm{Col}(A)$.
- If A is $n \times n$ then T does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation

Typical exam questions

- What does the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ do to $\mathbb{R}^3$?
- What does the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- True/false. If A is a matrix and T is the associated matrix transformation, then the statement $Ax = b$ is consistent is equivalent to the statement that b is in the range of T.
- True/false. There is a matrix A so that the domain of the associated matrix transformation is a line in $\mathbb{R}^3$.