Section 3.2

One-to-one and onto transformations

Section 3.2 Outline

- Learn the definitions of one-to-one and onto functions
- Determine if a given matrix transformation is one-to-one and/or onto

One-to-one and onto in calculus

What do one-to-one and onto mean for a function $f : \mathbb{R} \to \mathbb{R}$?

One-to-one

A matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

In other words: different inputs have different outputs.

Do not confuse this with the definition of a function, which says that for each input x in $\mathbb{R}^n$ there is at most one output b in $\mathbb{R}^m$.

One-to-one

$T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is one-to-one
- the columns of A are linearly independent
- $Ax = 0$ has only the trivial solution
- A has a pivot in each column
- the range of T has dimension n

What can we say about the relative sizes of m and n if T is one-to-one?

Draw a picture of the range of a one-to-one matrix transformation $\mathbb{R} \to \mathbb{R}^3$.

Onto

A matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

Onto

$T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is onto
- the columns of A span $\mathbb{R}^m$
- A has a pivot in each row
- $Ax = b$ is consistent for all b in $\mathbb{R}^m$
- the range of T has dimension m

What can we say about the relative sizes of m and n if T is onto?

Give an example of an onto matrix transformation $\mathbb{R}^3 \to \mathbb{R}$.

One-to-one and Onto

Do the following give matrix transformations that are one-to-one? onto?

$$\begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

One-to-one and Onto

Which of the previously-studied matrix transformations of $\mathbb{R}^2$ are one-to-one? Onto?

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ reflection

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projection

$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ scaling

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ shear

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotation

Which are one to one / onto?

Poll

Which give one to one-to-one / onto matrix transformations?

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

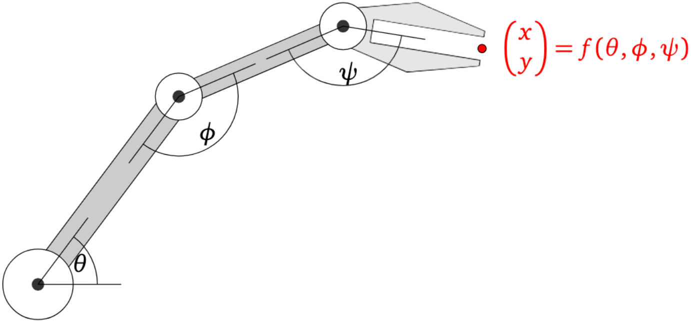
Demo

Demo

Demo

Robot arm

Consider the robot arm example from the book.



There is a natural function f here (not a matrix transformation). The input is a set of three angles and the co-domain is $\mathbb{R}^2$. Is this function one-to-one? Onto?

The geometry

Say that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

The geometry of one-to-one:

The range has dimension n (and the null space is a point).

The geometry of onto:

The range has dimension m, so it is all of $\mathbb{R}^m$ (and the null space has dimension $n - m$).

Summary of Section 3.2

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is one-to-one
  - the columns of A are linearly independent
  - $Ax = 0$ has only the trivial solution
  - A has a pivot in each column
  - the range has dimension n
- $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is onto
  - the columns of A span $\mathbb{R}^m$
  - A has a pivot in each row
  - $Ax = b$ is consistent for all b in $\mathbb{R}^m$.
  - the range of T has dimension m

Typical exam questions

- True/False. It is possible for the matrix transformation for a 5×6 matrix to be both one-to-one and onto.
- True/False. The matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by projection to the yz-plane is onto.
- True/False. The matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation by π is onto.
- Is there an onto matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not.
- Is there an one-to-one matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not.