Section 3.6

The invertible matrix theorem

Section 3.6 Outline

- The invertible matrix theorem

The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

(1) A is invertible
(2) T is invertible
(3) The reduced row echelon form of A is I_n
(4) A has n pivots
(5) $Ax = 0$ has only 0 solution
(6) $\mathrm{Nul}(A) = \{0\}$
(7) $\mathrm{nullity}(A) = 0$
(8) columns of A are linearly independent
(9) columns of A form a basis for $\mathbb{R}^n$
(10) T is one-to-one
(11) $Ax = b$ is consistent for all b in $\mathbb{R}^n$
(12) $Ax = b$ has a unique solution for all b in $\mathbb{R}^n$
(13) columns of A span $\mathbb{R}^n$
(14) $\mathrm{Col}(A) = \mathbb{R}^n$
(15) $\mathrm{rank}(A) = n$
(16) T is onto
(17) A has a left inverse
(18) A has a right inverse

The Invertible Matrix Theorem

There are two kinds of square matrices, invertible and non-invertible matrices.

For invertible matrices, all of the conditions in the IMT hold. And for a non-invertible matrix, all of them fail to hold.

One way to think about the theorem is: there are lots of conditions equivalent to a matrix having a pivot in every row, and lots of conditions equivalent to a matrix having a pivot in every column, and when the matrix is a square, all of these many conditions become equivalent.

Example

Determine whether A is invertible. $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$

It isn't necessary to find the inverse. Instead, we may use the Invertible Matrix Theorem by checking whether we can row reduce to obtain three pivot columns, or three pivot positions.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

There are three pivot positions, so A is invertible by the IMT (statement c).

The Invertible Matrix Theorem

Poll

Which are true? Why?

m) If A is invertible then the rows of A span $\mathbb{R}^n$

n) If $Ax = b$ has exactly one solution for all b in $\mathbb{R}^n$ then A is row equivalent to the identity.

o) If A is invertible then A^2 is invertible

p) If A^2 is invertible then A is invertible

Summary of Section 3.6

- Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.
  (1) A is invertible
  (2) T is invertible
  (3) The reduced row echelon form of A is I_n
  (4) etc.

More rabbits

Discussion Question

Recall that the following matrix describes the change in our rabbit population from this year to the next:

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Which of the following statements are true?

1. There is a population of rabbits that will result in 0 rabbits in the following year.
2. There are two different populations of rabbits that result in the same population in the following year
3. For any given population of rabbits, we can choose a population of rabbits for the current year that results in the given population in the following year (this is tricky!).

Typical Exam Questions Section 3.6

In all questions, suppose that A is an $n \times n$ matrix and that $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. For each question, answer YES or NO.

(1) Suppose that the reduced row echelon form of A does not have any zero rows. Must it be true that $Ax = b$ is consistent for all b in $\mathbb{R}^n$?

(2) Suppose that T is one-to-one. Is is possible that the columns of A add up to zero?

(3) Suppose that $Ax = e_1$ is not consistent. Is it possible that T is onto?

(4) Suppose that $n = 3$ and that $T \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 0$. Is it possible that T has exactly two pivots?

(5) Suppose that $n = 3$ and that T is one-to-one. Is it possible that the range of T is a plane?