Section 4.2

Cofactor expansions

Outline of Section 4.2

- We will give a recursive formula for the determinant of a square matrix.

A formula for the determinant

We will give a recursive formula.

First some terminology:

$$\begin{aligned} A_{ij} &= ij\text{th minor of } A \\ &= (n-1)\times(n-1) \text{ matrix obtained by deleting the } i\text{th row and } j\text{th column} \end{aligned}$$

$$\begin{aligned} C_{ij} &= (-1)^{i+j}\det(A_{ij}) \\ &= ij\text{th cofactor of } A \end{aligned}$$

Finally:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Or:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

So we find the determinant of a 3×3 matrix in terms of the determinants of 2×2 matrices, etc.

Determinants

Consider

$$A = \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

Compute the following:

$a_{11} =$ $a_{12} =$ $a_{13} =$

$A_{11} =$ $A_{12} =$ $A_{13} =$

$\det A_{11} =$ $\det A_{12} =$ $\det A_{13} =$

$C_{11} =$ $C_{12} =$ $C_{13} =$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

A formula for the determinant

We can take the recursive formula further....

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

Say that....

1×1 matrices

$$\det(a_{11}) = a_{11}$$

Now apply the formula to...

2×2 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =$$

(Could also go really nuts and define the determinant of a 0×0 matrix to be 1 and use the formula to get the formula for 1×1 matrices...)

A formula for the determinant

3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \cdots$$

You can write this out. And it is a good exercise. But you won't want to memorize it.

A formula for the determinant

Another formula for 3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

(Check this is gives the same answer as before. It is a small miracle!)

Use this formula to compute

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

Expanding across other rows and columns

The formula we gave for $\det(A)$ is the expansion across the first row. It turns out you can compute the determinant by expanding across any row or column:

$$\det(A) = a_{i1}C_{i1} + \cdots + a_{in}C_{in} \text{ for any fixed } i$$
$$\det(A) = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj} \text{ for any fixed } j$$

Or for odd rows and columns:

$$\det(A) = a_{i1}(\det(A_{i1})) - a_{i2}(\det(A_{i2})) + \cdots \pm a_{in}(\det(A_{in}))$$
$$\det(A) = a_{1j}(\det(A_{1j})) - a_{2j}(\det(A_{2j})) + \cdots \pm a_{nj}(\det(A_{nj}))$$

and for even rows and columns:

$$\det(A) = -a_{i1}(\det(A_{i1})) + a_{i2}(\det(A_{i2})) + \cdots \mp a_{in}(\det(A_{in}))$$
$$\det(A) = -a_{1j}(\det(A_{1j})) + a_{2j}(\det(A_{2j})) + \cdots \mp a_{nj}(\det(A_{nj}))$$

Amazingly, these are all the same!

Compute:

$$\det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

Another!

$$\det \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} =$$

Determinants of triangular matrices

If A is upper (or lower) triangular, $\det(A)$ is easy to compute with cofactor expansions (it was also easy using the definition of the determinant):

$$\det\begin{pmatrix} 2 & 1 & 5 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

Determinants

Poll

What is the determinant?

$$\det \begin{pmatrix} 4 & 7 & 0 & 9 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 5 & 9 & 2 & 10 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

A formula for the inverse

(from Section 3.3)

2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$n \times n$ matrices

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \dots & C_{nn} \end{pmatrix} \\ &= \frac{1}{\det(A)} (C_{ij})^T \end{aligned}$$

Check that these agree!

The proof uses Cramer's rule (see the notes on the course home page. We're not testing on this - it's just for your information.)

Summary of Section 4.2

- There is a recursive formula for the determinant of a square matrix:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

- We can use the same formula along any row/column.
- There are special formulas for the 2×2 and 3×3 cases.

Typical Exam Questions 4.2

- True or false. The cofactor expansion across the first row gives the negative of the cofactor expansion across the second row.
- Find the determinant of the following matrix using one of the formulas from this section:
$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & 0 & 9 \end{pmatrix}$$
- Find the determinant of the following matrix using one of the formulas from this section:
$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$$
- Find the cofactor matrix for the above matrix and use it to find the inverse.