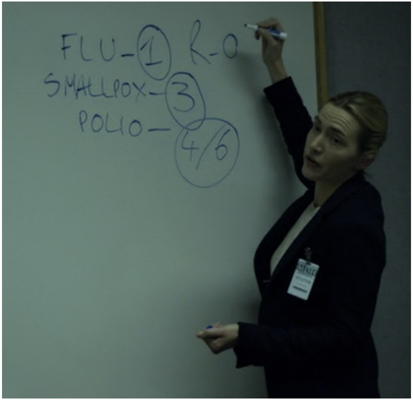
R_0

R_0

For a given virus, R_0 is the average number of people that each infected person infects. If R_0 is large, that is bad. Patient zero infects R_0 people, who then infect R_0^2 people, who then infect R_0^3 people. That is exponential growth. (If R_0 is less than 1, then that's good.)



R_0

For a given virus, R_0 is the average number of people that each infected person infects. If R_0 is large, that is bad. Patient zero infects R_0 people, who then infect R_0^2 people, who then infect R_0^3 people. That is exponential growth.

Whenever we see an exponential growth rate, we should think: eigenvalue.

It turns out that R_0 is an eigenvalue. The rough idea is very similar to our rabbit example: split the population into compartments, figure out how often each compartment infects each other compartment. That's a matrix. The largest eigenvalue is R_0.

R_0 is an eigenvalue

It turns out that R_0 is an eigenvalue. The rough idea is very similar to our rabbit example: split the population into compartments, figure out how often each compartment infects each other compartment.

For malaria, the compartments might be mosquitoes and humans.

For a sexually transmitted disease in a heterosexual population, the compartments might be males and females.

R_0 is an eigenvalue

It turns out that R_0 is an eigenvalue. The rough idea is very similar to our rabbit example: split the population into compartments, figure out how often each compartment infects each other compartment.

The SIR model has compartments for Susceptible, Infected, and Recovered.

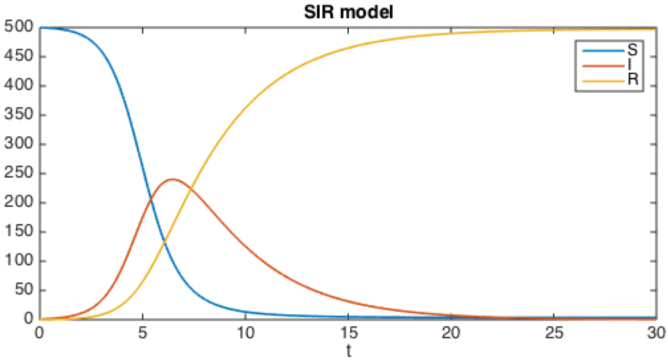
The arrows are governed by differential equations (Math 2552). Why do the labels on the arrows make sense? (The greek letters are constants).

There is a nice discussion of this by James Holland Jones (Stanford).

Paper

Bell curves

The growth rate of infection does not stay exponential forever, because the recovered population has immunity. That's where you get these bell curves.



Section 5.2

The characteristic polynomial

Outline of Section 5.2

- How to find the eigenvalues, via the characteristic polynomial
- Techniques for the 3×3 case

Characteristic polynomial

Recall:

$$\lambda \text{ is an eigenvalue of } A \iff A - \lambda I \text{ is not invertible}$$

So to find eigenvalues of A we solve

$$\det(A - \lambda I) = 0$$

The left hand side is a polynomial, the characteristic polynomial of A.

The roots of the characteristic polynomial are the eigenvalues of A.

The eigenrecipe

Say you are given a square matrix A.

Step 1. Find the eigenvalues of A by solving

$$\det(A - \lambda I) = 0$$

Step 2. For each eigenvalue λ_i the λ_i-eigenspace is the solution to

$$(A - \lambda_i I)x = 0$$

To find a basis, find the vector parametric solution, as usual.

Characteristic polynomial

Find the characteristic polynomial and eigenvalues of

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Two shortcuts for 2×2 eigenvectors

Find the eigenspaces for the eigenvalues on the last page. Two tricks.

(1) We do not need to row reduce $A - \lambda I$ by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

the eigenvector is

$$A = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Characteristic polynomial

Find the characteristic polynomial and eigenvalues of the Fibonacci matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

What does this tell you about Fibonacci numbers?

3×3 matrices

The 3×3 case is harder. There is a version of the quadratic formula for cubic polynomials, called Cardano's formula. But it is more complicated. It looks something like this:

$$\begin{aligned} x &= \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)+\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}} \\ &+ \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)-\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}. \end{aligned}$$

There is an even more complicated formula for quartic polynomials.

One of the most celebrated theorems in math, the Abel–Ruffini theorem, says that there is no such formula for quintic polynomials.

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ -3 & 0 & -1 \end{pmatrix}$$

What are the eigenvalues? Hint: Don't multiply everything out!

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ 4 & 2 & 0 \end{pmatrix}$$

Answer: $-\lambda^3 + 9\lambda^2 - 8\lambda$

What are the eigenvalues?

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Answer:

$$-\lambda^3 + 3\lambda + 2$$

What are the eigenvalues?

Hint: We already know one eigenvalue! Polynomial long division $\rightsquigarrow$

$$(\lambda - 2)(-\lambda^2 - 2\lambda - 1)$$

Don't really need long division: the first and last terms of the quadratic are easy to find; can guess and check the other term.

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial and eigenvalues.

$$\begin{pmatrix} 5 & -2 & 2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{pmatrix}$$

Characteristic polynomial: $-\lambda^3 + 9\lambda^2 - 23\lambda + 15$

This time we don't know any of the roots! We can use the rational root theorem: any integer root of a polynomial with leading coefficient ± 1 divides the constant term.

So we plug in ± 1, ± 3, ± 5, ± 15 into the polynomial and hope for the best. Luckily we find that 1, 3, and 5 are all roots, so we found all the eigenvalues!

If we were less lucky and found only one eigenvalue, we could again use long division like on the last slide.

Eigenvalues

Triangular matrices

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

Warning! You cannot find eigenvalues by row reducing and then using this fact. You need to work with the original matrix. Finding eigenspaces involves row reducing $A - \lambda I$, but there is no row reduction in finding eigenvalues.

Characteristic polynomials, trace, and determinant

The trace of a matrix is the sum of the diagonal entries.

The characteristic polynomial always looks like:

$$(-1)^n\lambda^n + (-1)^{n-1}\boxed{\text{trace(A)}}\lambda^{n-1} + \boxed{???}\lambda^{n-2} + \cdots \boxed{???}\lambda + \boxed{\det(A)}$$

So for a 2×2 matrix:

$$\lambda^2 - \text{trace}(A)\lambda + \det(A)$$

And for a 3×3 matrix:

$$-\lambda^3 + \text{trace}(A)\lambda^2 - \boxed{???}\lambda + \det(A)$$

Characteristic polynomials, trace, and determinant

The trace of a matrix is the sum of the diagonal entries.

The characteristic polynomial always looks like:

$$(-1)^n\lambda^n + (-1)^{n-1}\boxed{\text{trace(A)}}\lambda^{n-1} + \boxed{???}\lambda^{n-2} + \cdots \boxed{???}\lambda + \boxed{\det\text{(A)}}$$

Consequence 1. The constant term is zero $\Leftrightarrow$ A is not invertible

Consequence 2. The determinant is the product of the eigenvalues.

Algebraic multiplicity

The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example. Find the algebraic multiplicities of the eigenvalues for

$$\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Fact. The sum of the algebraic multiplicities of the (real) eigenvalues of an $n \times n$ matrix is at most n.

Summary of Section 5.2

- The characteristic polynomial of A is $\det(A - \lambda I)$
- The roots of the characteristic polynomial for A are the eigenvalues
- Techniques for 3×3 matrices:
  - Don't multiply out if there is a common factor
  - If there is no constant term then factor out λ
  - If the matrix is triangular, the eigenvalues are the diagonal entries
  - Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - Use the geometry to determine an eigenvalue
- Given an square matrix A:
  - The eigenvalues are the solutions to $\det(A - \lambda I) = 0$
  - Each λ_i-eigenspace is the solution to $(A - \lambda_i I)x = 0$

Typical Exam Questions 5.2

- True or false: Every $n \times n$ matrix has an eigenvalue.
- True or false: Every $n \times n$ matrix has n distinct eigenvalues.
- True or false: The nullity of $A - \lambda I$ is the dimension of the λ-eigenspace.
- What are the eigenvalues for the standard matrix for a reflection?
- What are the eigenvalues and eigenvectors for the $n \times n$ zero matrix?
- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 0 \end{pmatrix}$$

- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & 2 \end{pmatrix}$$

Hint: All of the eigenvalues are integers. Use the rational root theorem to guess one of the eigenvalues, and then factor out a linear term.