

# Eigenvalues in Structural Engineering

Watch this video about the Tacoma Narrows bridge. [▶ Watch](#)

Here are some toy models. [▶ Check it out](#)

The masses move the most at their **natural frequencies**  $\omega$ . To find those, use the spring equation:  $mx'' = -kx \rightsquigarrow \sin(\omega t)$ .

With 3 springs and 2 equal masses, we get:

$$mx_1'' = -kx_1 + k(x_2 - x_1)$$

$$mx_2'' = -kx_2 + k(x_1 - x_2)$$

Guess a solution  $x_1(t) = A_1(\cos(\omega t) + i \sin(\omega t))$  and similar for  $x_2$ . Finding  $\omega$  reduces to finding **eigenvalues** of  $\begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix}$ .

Eigenvectors:  $(1, 1)$  &  $(1, -1)$  (in/out of phase) [▶ Details](#)

# Section 5.4

## Diagonalization

## Section 5.4 Outline

- Diagonalization
- Using diagonalization to take powers
- Algebraic versus geometric dimension

## We understand diagonal matrices

We completely understand what diagonal matrices do to  $\mathbb{R}^n$ . For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

We have seen that it is useful to take powers of matrices: for instance in computing rabbit populations.

If  $A$  is diagonal, powers of  $A$  are easy to compute. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{10} =$$

## Powers of matrices that are similar to diagonal ones

What if  $A$  is not diagonal? Suppose want to understand the matrix

$$A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$$

geometrically? Or take it's 10th power? What would we do?

What if I give you the following equality:

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$A \qquad = \qquad C \qquad D \qquad C^{-1}$

This is called **diagonalization**.

How does this help us understand  $A$ ? Or find  $A^{10}$ ?

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This is called **diagonalization**.

How does this help us understand  $A$ ? Or find  $A^{10}$ ? [▶ Demo](#)

# Diagonalization

Suppose  $A$  is  $n \times n$ . We say that  $A$  is **diagonalizable** if we can write:

$$A = CDC^{-1} \quad D = \text{diagonal}$$

We say that  $A$  is similar to  $D$ .

How does this factorization of  $A$  help describe what  $A$  **does** to  $\mathbb{R}^n$ ?  
How does this help us take powers of  $A$ ?

Understanding the rabbit example: since 2 is the largest eigenvalue, (almost) all other vectors get pulled towards that eigenvector. Compare with the example from the last slide.

# Diagonalization

## The recipe

**Theorem.**  $A$  is diagonalizable  $\Leftrightarrow A$  has  $n$  linearly independent eigenvectors.

In this case

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_n & \end{pmatrix} \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}^{-1}$$
$$= \quad C \quad D \quad C^{-1}$$

where  $v_1, \dots, v_n$  are linearly independent eigenvectors and  $\lambda_1, \dots, \lambda_n$  are the corresponding eigenvalues, with multiplicity, in **order**.

Why?



## Example

Diagonalize if possible.

$$\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$$

## Example

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

## Example

Diagonalize if possible.

$$\begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$$

▶ Demo

*Hint: the eigenvalues are 1 and 1/2*

## Fibonacci numbers

Diagonalize the matrix.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

*Eigenvalues are  $\varphi$  &  $-1/\varphi$ , with eigenvectors  $(\varphi, 1)$  &  $(-1/\varphi, 1)$*

Use this to give a formula for the  $n$ th Fibonacci number

## More Examples

Diagonalize if possible.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Hint: the eigenvalues (with multiplicity) are 3, -1, 1 and 2, 2, 1

# Poll

Poll

Which are diagonalizable?

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

## Distinct Eigenvalues

Fact. If  $A$  has  $n$  distinct eigenvalues, then  $A$  is diagonalizable.

Why?

# Non-Distinct Eigenvalues

Theorem. Suppose

- $A = n \times n$ , has eigenvalues  $\lambda_1, \dots, \lambda_k$
- $a_i =$  algebraic multiplicity of  $\lambda_i$
- $d_i =$  dimension of  $\lambda_i$  eigenspace (“geometric multiplicity”)

Then

1.  $1 \leq d_i \leq a_i$  for all  $i$
2.  $A$  is diagonalizable  $\Leftrightarrow \sum d_i = n$   
 $\Leftrightarrow \sum a_i = n$  and  $d_i = a_i$  for all  $i$

So the recipe for checking diagonalizability is:

- If there are not  $n$  eigenvalues with multiplicity, then stop.
- For each eigenvalue with alg. mult. greater than 1, check if the geometric multiplicity is equal to the algebraic multiplicity. If any of them are smaller, the matrix is not diagonalizable.
- Otherwise, the matrix is diagonalizable.



## More rabbits

Which ones are diagonalizable?

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 4 & 4 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

*Hint: the characteristic polynomials are  $-\lambda^3 + 3\lambda + 2$  and  $-\lambda^3 + 2\lambda + 1$  and both have rational roots.*

## Summary of Section 5.4

- $A$  is diagonalizable if  $A = CDC^{-1}$  where  $D$  is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If  $A = CDC^{-1}$  then  $A^k = CD^kC^{-1}$
- $A$  is diagonalizable  $\Leftrightarrow A$  has  $n$  linearly independent eigenvectors  $\Leftrightarrow$  the sum of the geometric dimensions of the eigenspaces is  $n$
- If  $A$  has  $n$  distinct eigenvalues it is diagonalizable

## Typical Exam Questions 5.4

- True or False. If  $A$  is a  $3 \times 3$  matrix with eigenvalues 0, 1, and 2, then  $A$  is diagonalizable.
- True or False. It is possible for an eigenspace to be 0-dimensional.
- True or False. Diagonalizable matrices are invertible.
- True or False. Diagonal matrices are diagonalizable.
- True or False. Upper triangular matrices are diagonalizable.
- Find the 100th power of  $\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$ .
- For each of the following matrices, diagonalize or show they are not diagonalizable:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$