Chapter 6

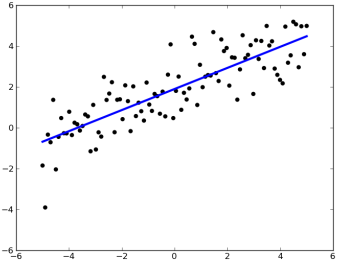
Orthogonality

Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



The answer relies on orthogonality.

Section 6.1

Dot products and Orthogonality

Outline

- Dot products
- Length and distance
- Orthogonality

Dot product

Say $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$

$$
\begin{aligned}
u \cdot v &= \sum_{i=1}^{n} u_i v_i \\
&= u_1 v_1 + \cdots + u_n v_n \\
&= u^T v
\end{aligned}
$$

Example. Find $(1, 2, 3) \cdot (4, 5, 6)$.

Dot product

Some properties of the dot product

- $u \cdot v = v \cdot u$
- $(u + v) \cdot w = u \cdot w + v \cdot w$
- $(cu) \cdot v = c(u \cdot v)$
- $u \cdot u \geq 0$
- $u \cdot u = 0 \Leftrightarrow u = 0$

Length

Let v be a vector in $\mathbb{R}^n$

$$\begin{aligned} \|v\| &= \sqrt{v \cdot v} \\ &= \text{length of } v \end{aligned}$$

Why? Pythagorean Theorem

Fact. $\|cv\| = |c|\|v\|$

v is a unit vector of $\|v\| = 1$

Problem. Find the unit vector in the direction of $(1, 2, 3, 4)$.

Distance

The distance between v and w is the length of $v - w$ (or $w - v$!).

Problem. Find the distance between $(1, 1, 1)$ and $(1, 4, -3)$.

Orthogonality

Fact. $u \perp v \Leftrightarrow u \cdot v = 0$

Why? Pythagorean theorem again!

$$\begin{aligned} u \perp v &\Leftrightarrow \|u\|^2 + \|v\|^2 = \|u - v\|^2 \\ &\Leftrightarrow u \cdot u + v \cdot v = u \cdot u - 2u \cdot v + v \cdot v \\ &\Leftrightarrow u \cdot v = 0 \end{aligned}$$

Problem. Find a vector in $\mathbb{R}^3$ orthogonal to $(1, 2, 3)$.

Summary of Section 6.1

- $u \cdot v = \sum u_i v_i$
- $u \cdot u = \|u\|^2$ (length of u squared)
- The unit vector in the direction of v is $v/\|v\|$.
- The distance from u to v is $\|u - v\|$
- $u \cdot v = 0 \Leftrightarrow u \perp v$