Section 6.2

Orthogonal complements

Outline of Section 6.2

- Orthogonal complements
- Computing orthogonal complements

Orthogonal complements

W = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$?
What about the orthogonal complement of a plane in $\mathbb{R}^3$?

Demo Demo

Orthogonal complements

W = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$ (it's a null space!)
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$ (rank-nullity theorem!)
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
   $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
5. The intersection of W and $W^\perp$ is $\{0\}$.

For items 1 and 3, which linear transformation do we use?

Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1, 1, -1)\}$. Find the equation of the plane $W^\perp$.

Find a basis for $W^\perp$.

Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1,1,-1),(-1,2,1)\}$. Find a system of equations describing the line $W^{\perp}$.

Find a basis for $W^{\perp}$.

Orthogonal complements

Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

Orthogonal complements

Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

In other words:

Theorem. $A = m \times n$ matrix

$$(\mathrm{Row} A)^\perp = \mathrm{Nul}\, A$$

Geometry $\leftrightarrow$ Algebra

(The row space of A is the span of the rows of A.)

Orthogonal decomposition

Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where v_W is in W and $v_{W^\perp}$ is in $W^\perp$.

Why?

Demo

Demo

Next time: Find v_W and $v_{W^\perp}$.

Orthogonal decomposition

Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where v_W is in W and $v_{W^\perp}$ is in $W^\perp$.

Why? Say that $w_1 + w_1' = w_2 + w_2'$ where w_1 and w_2 are in W and w_1' and w_2' are in $W^\perp$. Then $w_1 - w_2 = w_2' - w_1'$. But the former is in W and the latter is in $W^\perp$, so they must both be equal to 0.

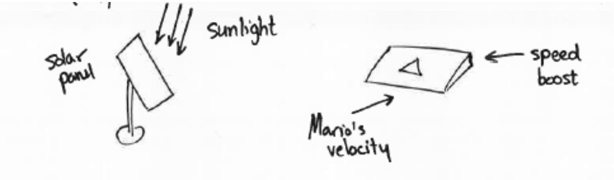
▸ Demo

▸ Demo

Next time: Find v_W and $v_{W^\perp}$.

Orthogonal Projections

Many applications, including:



Summary of Section 6.2

- $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$
- Facts:
  1. $W^\perp$ is a subspace of $\mathbb{R}^n$
  2. $(W^\perp)^\perp = W$
  3. $\dim W + \dim W^\perp = n$
  4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
  5. The intersection of W and $W^\perp$ is $\{0\}$.
- To find $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find the null space.
- Every vector v can be written uniquely as $v = v_W + v_{W^\perp}$ with v_W in W and $v_{W^\perp}$ in $W^\perp$

Typical Exam Questions 6.2

- What is the dimension of $W^\perp$ if W is a line in $\mathbb{R}^{10}$?
- What is $W^\perp$ if W is the line $y = mx$ in $\mathbb{R}^2$?
- If W is the x-axis in $\mathbb{R}^2$, and $v = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, write v as $v_W + v_{W^\perp}$.
- If W is the line $y = x$ in $\mathbb{R}^2$, and $v = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, write v as $v_W + v_{W^\perp}$.
- Find a basis for the orthogonal complement of the line through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$.
- Find a basis for the orthogonal complement of the line through $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ in $\mathbb{R}^4$.
- What is the orthogonal complement of x_1x_2-plane in $\mathbb{R}^4$?