1. We defined the dimension of a subspace $V$ to be the number of vectors in a basis for $V$. There's one problem: we haven't shown that all bases have the same number of vectors! The goal of this exercise is to explain why any two bases for $V$ must have the same number of vectors.

Suppose $\{b_1, \ldots, b_k\}$ is a basis for the subspace $V$ of $\mathbb{R}^n$. Let $\{a_1, \ldots, a_\ell\}$ be a set of vectors in $V$ with $\ell > k$. We want to show that $\{a_1, \ldots, a_\ell\}$ is not a basis for $V$ and we will do this by showing that $\{a_1, \ldots, a_\ell\}$ is linearly dependent.

Let $A$ be the matrix $(a_1 \cdots a_\ell)$ and let $B$ be the matrix $(b_1 \cdots b_k)$.

**Step 1.** For each $a_i$ in $A$, explain why there is a vector $c_i$ in $\mathbb{R}^k$ so that $Bc_i = a_i$. **Hint:** think about converting vector equations to matrix equations.

If $\vec{a}_i$ is in $V$, then $\vec{a}_i$ is a linear combination of the basis $\{\vec{b}_1, \ldots, \vec{b}_k\}$.

Thus, $\vec{a}_i = c_{i,1}\vec{b}_1 + c_{i,2}\vec{b}_2 + \ldots + c_{i,k}\vec{b}_k$.

$\Rightarrow \vec{a}_i = \begin{bmatrix} c_{i,1} \\ c_{i,2} \\ \vdots \\ c_{i,k} \end{bmatrix} = B \begin{bmatrix} c_{i,1} \\ c_{i,2} \\ \vdots \\ c_{i,k} \end{bmatrix}$

Now let $C$ be the matrix $(c_1 \cdots c_k)$.

**Step 2.** Explain why $Cx = 0$ has a nonzero solution. **Hint:** use the fact that $k < \ell$.

Recall that each vector $\vec{c}_i$ is in $\mathbb{R}^k$.

Matrix $C$ thus has $k$ rows.

Matrix $C$ also has $\ell$ columns, and $\ell > k$.

If a matrix has more columns than rows, it can't have a pivot in every column.

$\Rightarrow Cx = 0$ has nontrivial solution.
We have shown that no basis can have more vectors than another basis. Thus, all bases have the same number of vectors.

Now let $\tilde{u} = (u_1, \ldots, u_k)$ be a nonzero solution to $Cx = 0$.

Step 3. Show that $Au = 0$. Hint: Write $Au$ as a linear combination of the $a_i$ and then replace each $a_i$ in the vector equation with $Bc_i$ and then factor out the $B$.

$$A\tilde{u} = u_1 a_1 + u_2 a_2 + \ldots + u_k a_k$$
$$= u_1 Bc_1 + u_2 Bc_2 + \ldots + u_k Bc_k$$
$$= B[u_1 c_1 + u_2 c_2 + \ldots + u_k c_k]$$
$$= B[\tilde{c}]$$
$$= B \cdot 0$$
$$= 0$$

Thus, $Au = 0$.

Step 4. Conclude that $\{a_1, \ldots, a_k\}$ is linearly dependent and that any two bases for $V$ have the same number of elements.

Since $Ax = 0$ has a nontrivial solution (see above), then the columns of $A$, or rather $\{a_1, \ldots, a_k\}$, are linearly dependent.

Thus, if $k \leq n$, the linear independence requirement for a basis is not met.

We have shown that no basis can have more vectors than another basis. Thus, all bases have the same number of vectors.