$Ad_i = b_i$

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Name Solution

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Mathematics 1553

Written Homework 6
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1. We defined the dimension of a subspace V to be the number of vectors in a basis for V. There's one problem: we haven't shown that all bases have the same number of vectors! The goal of this exercise is to explain why any two bases for V must have the same number of vectors.

Suppose $\{b_1, \ldots, b_k\}$ is a basis for the subspace V of $\mathbb{R}^n$. Let $\{a_1, \ldots, a_\ell\}$ be a set of vectors in V with $\ell > k$. We want to show that $\{a_1, \ldots, a_\ell\}$ is not a basis for V and we will do this by showing that $\{a_1, \ldots, a_\ell\}$ is linearly dependent.

Let A be the matrix $(a_1 \cdots a_\ell)$ and let B be the matrix $(b_1 \cdots b_k)$.

Step 1. For each a_i in A, explain why there is a vector c_i in $\mathbb{R}^k$ so that $Bc_i = a_i$. *Hint: think about converting vector equations to matrix equations.*

If $\vec{a_i}$ is in V, then $\vec{a_i}$ is a linear combination of the basis $\{\vec{b_1}, \ldots, \vec{b_k}\}$

$\Rightarrow$ Thus, $\vec{a_i} = c_{i1}\vec{b_1} + c_{i2}\vec{b_2} + \ldots + c_{ik}\vec{b_k}$

$$\Rightarrow \vec{a_i} = \underbrace{\begin{bmatrix} \vec{b_1} & \vec{b_2} & \ldots & \vec{b_k} \end{bmatrix}}_{B} \begin{bmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{ik} \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{a_i} = B\vec{c_i}}$$

Now let C be the matrix $(c_1 \cdots c_\ell)$.

Step 2. Explain why $Cx = 0$ has a nonzero solution. *Hint: use the fact that $k < \ell$.*

Recall that each vector $\vec{c_i}$ is in $\mathbb{R}^k$

Matrix C thus has k rows

Matrix C also has l columns, and $l > k$

If a matrix has more columns than rows, it can't have a pivot in every column

$\Rightarrow$ $\boxed{C\vec{x} = 0 \text{ has nontrivial solution}}$

Now let $\vec{u} = (u_1, \ldots, u_\ell)$ be a nonzero solution to $Cx = 0$.

Step 3. Show that $Au = 0$. *Hint: Write Au as a linear combination of the a_i and then replace each a_i in the vector equation with Bc_i and then factor out the B.*

$$\begin{aligned}
A\vec{u} &= u_1\vec{a}_1 + u_2\vec{a}_2 + \ldots + u_\ell\vec{a}_\ell \\
&= u_1B\vec{c}_1 + u_2B\vec{c}_2 + \ldots + u_\ell B\vec{c}_\ell \\
&= B[u_1\vec{c}_1 + u_2\vec{c}_2 + \ldots + u_\ell\vec{c}_\ell] \\
&= B[C\vec{u}] \\
&= B \cdot 0 \\
&= \boxed{0} \checkmark
\end{aligned}$$

Step 4. Conclude that $\{a_1, \ldots, a_\ell\}$ is linearly dependent and that any two bases for V have the same number of elements.

Since $A\vec{x} = 0$ has a nontrivial solution (see above), then the columns of A, or rather $\{a_1, \ldots, a_\ell\}$, are linearly dependent

$\Rightarrow$ Thus, if $\ell > k$, the linear independence requirement for a basis is not met.

We have shown that no basis can have more vectors than another basis. Thus, all bases have the same number of vectors.