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Section H J
Subsection left center right

Mathematics 1553

Written Homework 8
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1. The goal of this assignment is to find a formula for the nth Fibonacci number. You might have seen the Fibonacci numbers before:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

After the first two numbers, each number in the sequence is the sum of the previous two. In other words we have a *recursion relation*:

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2} \quad n \geq 2 \end{aligned}$$

Problem. Find a formula for the nth Fibonacci number f_n.

The first thing we'll show is that we can get all the Fibonacci numbers from the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and the vector corresponding to the first two Fibonacci numbers

$$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

What is Ae_2?

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What is A^2e_2?

$$A^2\vec{e_2} = A(Ae_2) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

What is A^3e_2?

$$A^3\vec{e_2} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

What is A^ne_2? Your answer should be in terms of the Fibonacci numbers $f_0, f_1, f_2, \ldots$.

$$A^n e_2 = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$$

Based on the answer to the last question, you can find a formula for the nth Fibonacci number if you can find the powers of A. To do this we will want to diagonalize A.

Find the eigenvalues of A.

$$A-\lambda I = \begin{pmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$$

$$\det[A-\lambda I] = -\lambda\cdot(1-\lambda) - 1 = \lambda^2 - \lambda - 1$$

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\cdot 1\cdot 1}}{2\cdot 1} = \frac{1 \pm \sqrt{5}}{2}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2}$$

You can make your calculations later easier by naming the larger eigenvalue a and the smaller one $-1/a$ (it so happens that for this matrix the eigenvalues are negative reciprocals). The number a is a famous number called the golden ratio.

Find an eigenvector for $\lambda = a$. Your answer will be in terms of a. *Hint: You know that the eigenspace must be 1-dimensional. So the second row of $A - \lambda I$ is a multiple of the first and you can replace the second row with a row of zeros.*

$$A - \lambda I = \begin{pmatrix} -a & 1 \\ 1 & 1-a \end{pmatrix} \rightsquigarrow \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -a & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} -ax_1 + x_2 = 0 \\ x_2 \text{ free} \end{cases} \quad \text{pick } x_2 = a.$$

$$\Rightarrow \vec{V}_1 = \begin{pmatrix} 1 \\ a \end{pmatrix}$$

Find an eigenvector for $\lambda = -1/a$. Your answer will again be in terms of a.

$$A - \lambda_2 I = \begin{pmatrix} +\frac{1}{a} & 1 \\ 1 & 1+\frac{1}{a} \end{pmatrix} \rightsquigarrow \begin{pmatrix} +\frac{1}{a} & 1 \\ 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} +\frac{1}{a} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} +\frac{1}{a}x_1 + x_2 = 0 \\ x_2 \text{ free} \end{cases} \quad \text{pick } x_2 = -\frac{1}{a} \qquad \vec{V}_2 = \begin{pmatrix} 1 \\ -\frac{1}{a} \end{pmatrix}$$

Diagonalize A. Your answers on this page will again be in terms of a.

$$A = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ a & -\frac{1}{a} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & -\frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ a & -\frac{1}{a} \end{pmatrix}^{-1} =$$

$$= CDC^{-1}$$

Use your diagonalization of A to find a formula for the nth power of A.

$$A^n = (CDC^{-1})^n = \underbrace{(CD\cancel{C^{-1}}) \cdot (\cancel{C}D\cancel{C^{-1}}) \cdots (\cancel{C}DC^{-1})}_{n \text{ groups of } (CDC^{-1})}$$

$$= CD^nC^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ a & -\frac{1}{a} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & -\frac{1}{a} \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ a & -\frac{1}{a} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ a & -\frac{1}{a} \end{pmatrix} \begin{pmatrix} a^n & 0 \\ 0 & (-\frac{1}{a})^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ a & -\frac{1}{a} \end{pmatrix}^{-1}$$

Use your formula for A^n to find a formula for f_n (*hint: multiply A^n by e_2*).

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = A^n \vec{e_2} \Rightarrow f_n = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} A^n \vec{e_2}$$

1×2 2×2

$$\Rightarrow f_n = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ a & -\frac{1}{a} \end{pmatrix} \begin{pmatrix} a^n & 0 \\ 0 & (-\frac{1}{a})^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ a & -\frac{1}{a} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} a^n & 0 \\ 0 & (-\frac{1}{a})^n \end{pmatrix} \cdot \frac{1}{1 \cdot (-\frac{1}{a}) - a} \begin{pmatrix} -\frac{1}{a} & -1 \\ -a & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= + \frac{a}{1+a^2} \begin{pmatrix} a^n & (-\frac{1}{a})^n \end{pmatrix} \cdot \begin{pmatrix} +1 \\ -1 \end{pmatrix} = + \frac{a}{1+a^2} \left(a^n - (-\tfrac{1}{a})^n\right)$$