1. In the following problems, A is an $m \times n$ matrix (m rows and n columns) and b is a vector in $\mathbb{R}^m$. Let T_A be the linear transformation associated to A. Circle TRUE if the statement is always true (for any choices of A and b) and circle FALSE otherwise. Do not assume anything else about A or b except what is stated.

The matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is in reduced row echelon form.

TRUE FALSE

If A has fewer than m pivots then $Ax = b$ has infinitely many solutions.

TRUE FALSE

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If $m < n$ then the columns of A are linearly dependent.

TRUE FALSE

$m<n \Rightarrow$ At most m pivots $\Rightarrow$ not all columns have pivot.

$\Rightarrow$ columns are L.D.

The zero vector is in the range of T_A.

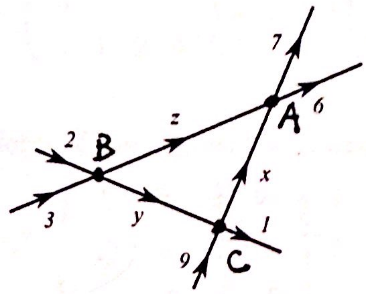
TRUE FALSE

$A\vec{0} = \vec{0}$

If $Ax = b$ is consistent then b is in the span of the columns of A.

TRUE FALSE

2. The following diagram indicates traffic flow in one part of town (the numbers indicate the number of cars per minute on each section of road):



Write a system of linear equations in x, y, and z describing the traffic flow around the triangle.

$$\begin{cases} x+z=7+6 & \text{point A} \\ y+z=2+3 & \text{point B} \\ y+9=x+1 & \text{point C} \end{cases} \Rightarrow \begin{cases} x+z=13 \\ y+z=5 \\ -x+y=-8 \end{cases}$$

Write the above system of linear equations as a vector equation (recall that a vector equation would look like a variable times a vector plus another variable times a vector, etc.). Do not solve the vector equation.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \\ -8 \end{pmatrix}$$

3. Consider the matrix equation $Ax = b$ where

$$A = \begin{pmatrix} 1 & 3 & 8 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Find the reduced row echelon form of the augmented matrix $(A \mid b)$.

$$\left(\begin{array}{cccc|c} 1 & 3 & 8 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 4 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 3 & 8 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Write the set of solutions to $Ax = b$ in parametric form.

$$\begin{cases} x_1 = -2 - 2x_3 \\ x_2 = 1 - 2x_3 \\ x_3 = x_3 \\ x_4 = 0 \end{cases} \Rightarrow \vec{x} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

What best describes the geometric relationship between the solutions to $Ax = 0$ and the solutions to $Ax = b$ (same A and b as above)?

(a) they are both lines through the origin

(b) they are parallel lines   1 degree of freedom.

(c) they are both planes through the origin

(d) they are parallel planes

4. Find all values of k so that the following set of vectors is linearly dependent.

$$\left\{ \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ k \\ -7 \end{pmatrix} \right\}$$

$$\begin{pmatrix} -1 & 1 & 1 \\ 3 & 1 & k \\ -1 & -1 & -7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 4 & k+3 \\ 0 & 2 & 8 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 4 & k+3 \\ 0 & 0 & k-13 \end{pmatrix}$$

linearly dependent => less pivots than columns

$\Rightarrow k-13=0$

$\Rightarrow \boxed{k=13}$

5. *(a)* Consider the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

and let T_A be the associated linear transformation.

Is T_A one-to-one?

Yes, since each column of A has a pivot.

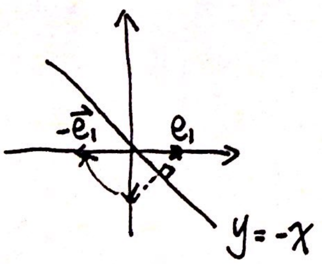
Find one nonzero vector b in the range of T_A.

$\vec{b} = A\vec{x}$ pick $\vec{x} = \begin{pmatrix}1\\1\end{pmatrix}$

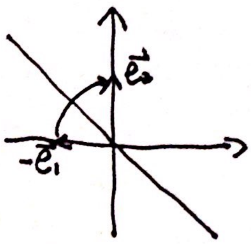
$\vec{b} = \begin{pmatrix}-1\\2\end{pmatrix}$

(b) Find a matrix A so that T_A is the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ obtained by first reflecting about the line $y = -x$ and then rotating clockwise by $\pi/2$. (Note: this problem is completely independent of the first problem on this page—the two As have nothing to do with each other.)

$T_A(e_1)$:



$T(e_2)$



$A = \begin{pmatrix} T_A(\vec{e_1}) & T_A(\vec{e_2}) \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

After reflecting : $\vec{e_1} \to -\vec{e_2}$ $\qquad \vec{e_2} \Rightarrow -\vec{e_1}$

After rotation : $-\vec{e_2} \Rightarrow -\vec{e_1}$ $\qquad -\vec{e_1} \to \vec{e_2}$

$\Rightarrow T_A(\vec{e_1}) = -\vec{e_1}$ $\qquad \Rightarrow T_A(\vec{e_2}) = \vec{e_2}$