1. Answer each of the following questions. You do not need to explain your answer.

Assume that A is an $n \times n$ matrix, v is a vector in $\mathbb{R}^n$ and λ is a real number. Which of the following correctly characterizes v and λ as an eigenvector and eigenvalue for A?

(a) $Av = \lambda v$

(b) $Av = \lambda v$ and $\lambda \neq 0$

(c) $Av = \lambda v$ and $v \neq 0$

(d) $Av = \lambda v$, $v \neq 0$, and $\lambda \neq 0$

If A is a 3×3 real-value matrix, then A always has at least one real eigenvalue.

TRUE FALSE

For a matrix that has all real entries, if there are complex eigenvalues, they comes in pairs. So for 3×3 matrix, one can at most have 1 pair of complex eigenvalues, and thus at least 1 real eigenvalue.

If two matrices have the same eigenvalues (with the same algebraic multiplicities) then they are similar.

TRUE FALSE

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ both have eigenvalue of 1

$(1-\lambda)^2 = 0$ $(1-\lambda)^2 - 0 = 0$

Find a 2×2 matrix that is neither invertible nor diagonalizable. *Hint: use 1's and 0's.*

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

2. Answer each of the following questions. You do not need to explain your answer.

If A is a 3×3 matrix and its characteristic polynomial is $\lambda^3 + \lambda^2 + \lambda$, then A is invertible.

YES NO MAYBE

one of the roots for λ is $\lambda = 0$. $\Rightarrow$ $A\vec{x} = \vec{0}$ has non-trivial solution

$\Rightarrow$ A is not invertible.

A 3×3 matrix has two distinct eigenvalues. Is A diagonalizable?

YES NO MAYBE

Diagonalizable if has 3 L.I. eigen vectors.

If A is diagonalizable and B is similar to A, is B diagonalizable?

YES NO MAYBE

B similar to A $\Rightarrow$ $B = CAC^{-1}$

A diagonalizable $\Rightarrow$ $A = PDP^{-1}$

$\Rightarrow$ $B = CPDP^{-1}C^{-1} = (CP)D(CP)^{-1}$

Suppose that A is a 2×2 matrix and the associated linear transformation T_A is reflection about the line $y = 5x$. What are the eigenvalues of A?

$\lambda = 1$ or -1

if the vector lies on $y = 5x$ ie $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$, after reflection, the vector would not change magnitude nor direction $\Rightarrow$ $\lambda = 1$.

if the vector is perpendicular to $y = 5x$ ie $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$ after reflection, the vector would have an opposite direction but of the same magnitude. $\Rightarrow$ $\lambda = -1$

3. Consider the following matrix.

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 0 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}$$

Find the eigenvalues of A.

$$A-\lambda I = \begin{pmatrix} 4-\lambda & 2 & -4 \\ 0 & 2-\lambda & 0 \\ 2 & 2 & -2-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (2-\lambda)\begin{vmatrix} 4-\lambda & -4 \\ 2 & -2-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2-2\lambda \quad) = -\lambda(\lambda-2)^2 = 0 \quad \lambda-2$$

$\lambda_1 = 2$ with multiplicity of 2; $\lambda_2 = 0$ with multiplicity of 1

For each of the eigenvalues of A, find a basis for the corresponding eigenspace.

for $\lambda = 2$

$$A\vec{x} = 2\vec{x} \Rightarrow \left(\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \vec{x} = x_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

for $\lambda=2$ eigenspace: $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right\}$

for $\lambda = 0$

$$A\vec{x} = \vec{0} \Rightarrow \left(\begin{array}{ccc|c} 4 & 2 & -4 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 2 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \vec{x} = x_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

eigen space for $\lambda=0$: $\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$

Is A diagonalizable? If so, diagonalize it. If not, explain why not.

Yes:

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$$

4. Consider the matrix

$$A = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}$$

Find the (complex) eigenvalues of A.

$$(3-\lambda)(-3-\lambda)+10 = \lambda^2+1 = 0$$

$$\lambda = \pm j$$

For each of the eigenvalues of A, find an eigenvector.

for $\lambda_1 = j$ $\quad (A-\lambda I \mid 0) \Rightarrow \left(\begin{array}{cc|c} 3-j & -5 & 0 \\ 2 & -3-j & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 3-j & -5 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$V_1 = \begin{pmatrix} 5 \\ 3-j \end{pmatrix}$$

for $\lambda_2 = -j$ $\quad \left(\begin{array}{cc|c} 3+j & -5 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$V_2 = \begin{pmatrix} 5 \\ 3-j \end{pmatrix}$$

Find a rotation-plus-scaling matrix B that is similar to A.

For a pure rotation matrix $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ the eigenvalues are $\cos\theta \pm j\sin\theta$

As B similar to A, they have same eigenvalues $\Rightarrow \cos\theta = 0$; $\sin\theta = 1$

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

By how much does B scale?

Since $\det B = 1$, so B scale by a factor of 1

By what angle does B rotate?

Since $\sin\theta = 1 \Rightarrow \theta$ could be 90°

5. Consider the sequence of numbers $0, 1, 5, 31, 185, \ldots$ given by

$$a_0 = 0$$
$$a_1 = 1$$
$$a_n = 5a_{n-1} + 6a_{n-2} \quad n \geq 2$$

Consider also

$$A = \begin{pmatrix} 0 & 1 \\ 6 & 5 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can diagonalize A as

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 6 \end{pmatrix}^{-1}$$

Give a formula for A^n. Your answer should be a single matrix.

$$A^n = \begin{pmatrix} 1 & 1 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 6^n \end{pmatrix} \left\{ \begin{pmatrix} 6 & -1 \\ 1 & 1 \end{pmatrix} \cdot \frac{1}{7} \right\}$$

$$= \begin{pmatrix} (-1)^n & 6^n \\ (-1)^{n+1} & 6^{n+1} \end{pmatrix} \begin{pmatrix} \frac{6}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix} = \begin{pmatrix} (-1)^n \cdot \frac{6}{7} + \frac{6^n}{7} & \frac{(-1)^{n+1}}{7} + \frac{6^n}{7} \\ (-1)^{n+1} \cdot \frac{6}{7} + \frac{6^{n+1}}{7} & \frac{(-1)^n}{7} + \frac{6^{n+1}}{7} \end{pmatrix}$$

Multiple choice. What is $A^n e_2$? (Note that the sequence above starts with a_0.)

$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} \qquad \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix} \qquad \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} \qquad \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$$

Use your previous two answers to give a formula for a_n.

$$A^n e_2 = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} \quad \Rightarrow \quad a_n = (1 \;\; 0) A^n e_2$$

$$a_n = \frac{(-1)^{n+1}}{7} + \frac{6^n}{7}$$