Announcements April 11

Quiz Fri

- WebWork 6.1 and 6.2 due Thursday
- Final Exam Wed May 4 8:00-10:50 (Sec H) and Mon May 2 2:50-5:40 (Sec J)
- ~~Tell me now if you have a conflict (three exams in one day, Math 1553 in middle)~~
- Office Hours Tue 2-3 and Wed 2-3
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Chapter 6

Orthogonality and Least Squares

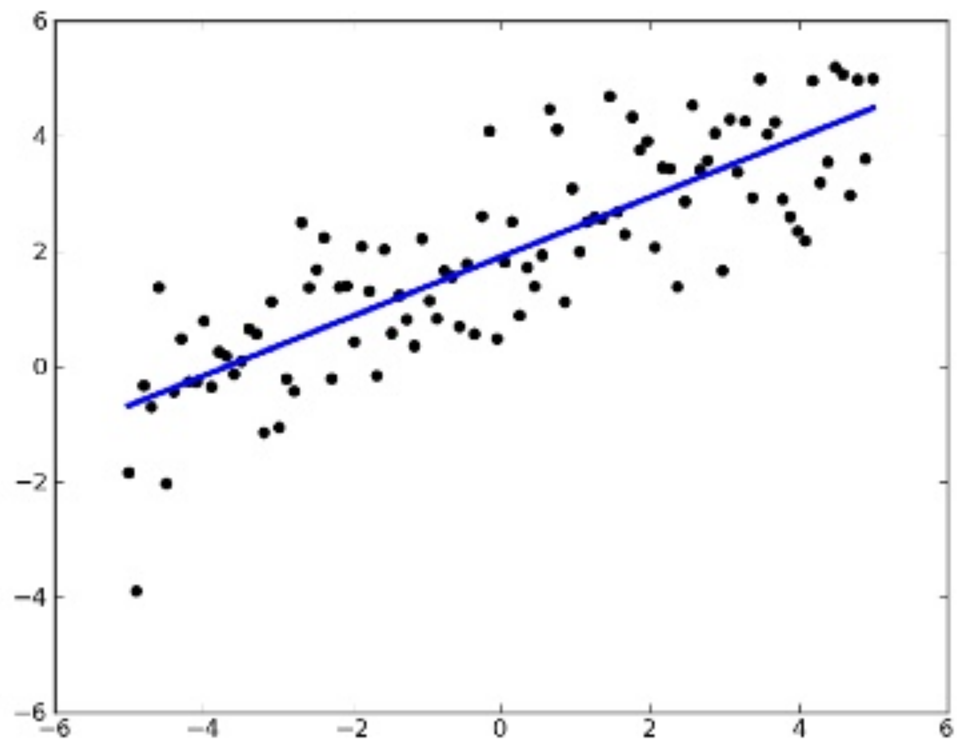
Section 6.1

Inner Product, Length, and Orthogonality

Where are we?

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



Solve $Ax=b$ as close as possible.

The answer relies on orthogonality.

Outline

- Dot products
- Dot products and orthogonality
- Orthogonal projection
- A formula for projection onto a line
- Orthogonal complements

orthogonal = perpendicular

Dot product

Say $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$

$$\begin{aligned} u \cdot v &= \sum_{i=1}^{n} u_i v_i \\ &= u_1 v_1 + \cdots + u_n v_n \\ &= u^T v \end{aligned}$$

thinking of u, v as col vectors

$$\underset{u^T}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}} \underset{v}{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}} = \begin{pmatrix} 32 \end{pmatrix}$$

Example. Find $(1, 2, 3) \cdot (4, 5, 6)$.

$$= 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6$$

$$= 32$$

Dot product

Some properties of the dot product

- $u \cdot v =$ $v \cdot u$
- $(u + v) \cdot w =$ $u \cdot w + v \cdot w$
- $(cu) \cdot v =$ $c\,(u \cdot v)$
- $u \cdot u$ ≥ 0
- $u \cdot u = 0 \Leftrightarrow$ $u = 0$

$(1,0,0) \cdot (1,0,0)$

$= 1$

$(1,0,0) \cdot (0,1,0)$

$= 0$

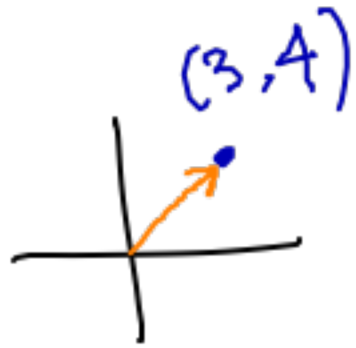
Dot product
and Length

Let v be a vector in $\mathbb{R}^n$

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2+v_2^2+\cdots+v_n^2} \quad \text{if } v=(v_1,\ldots,v_n)$$

$= \text{length (or norm) of } v$

Why?



$$\|(3,4)\| = \sqrt{3^2+4^2} = 5$$

Fact. $\|cv\| = c\|v\|$ $\quad \|(6,8)\| = 2\cdot 5 = 10.$

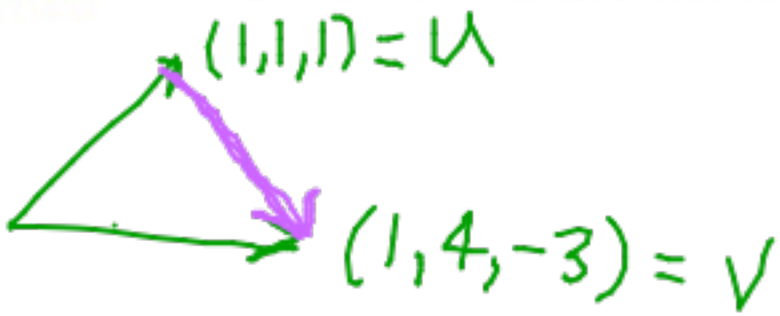
v is a unit vector of $\|v\|=1$

Problem. Find the unit vector in the direction of $(1,2,3,4)$.

$$\|(1,2,3,4)\| = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{30}$$

answer: $\left(1/\sqrt{30},\ 2/\sqrt{30},\ 3/\sqrt{30},\ 4/\sqrt{30}\right)$ $\quad (1,2,3,4)$

Problem. Find the distance between $(1,1,1)$ and $(1,4,-3)$.



$$v-u = (0,3,-4)$$

$$\|v-u\| = \sqrt{3^2+(-4)^2} = 5$$

Orthogonality

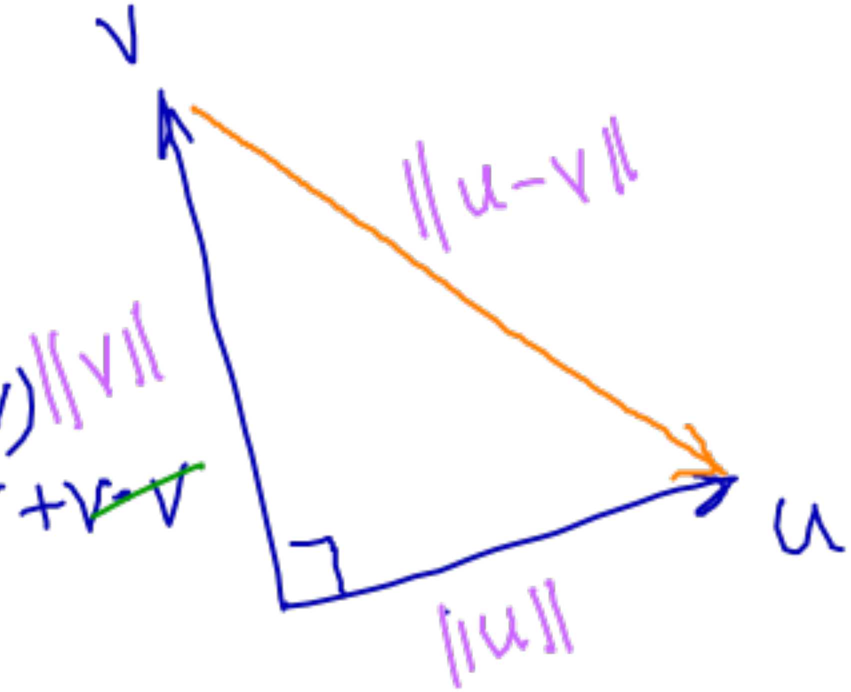
Fact. $u \perp v \Leftrightarrow u \cdot v = 0$

Why? $u \perp v \iff \|u\|^2 + \|v\|^2 = \|u-v\|^2$

$$\iff u\cdot u + v \cdot v = (u-v)\cdot(u-v)$$

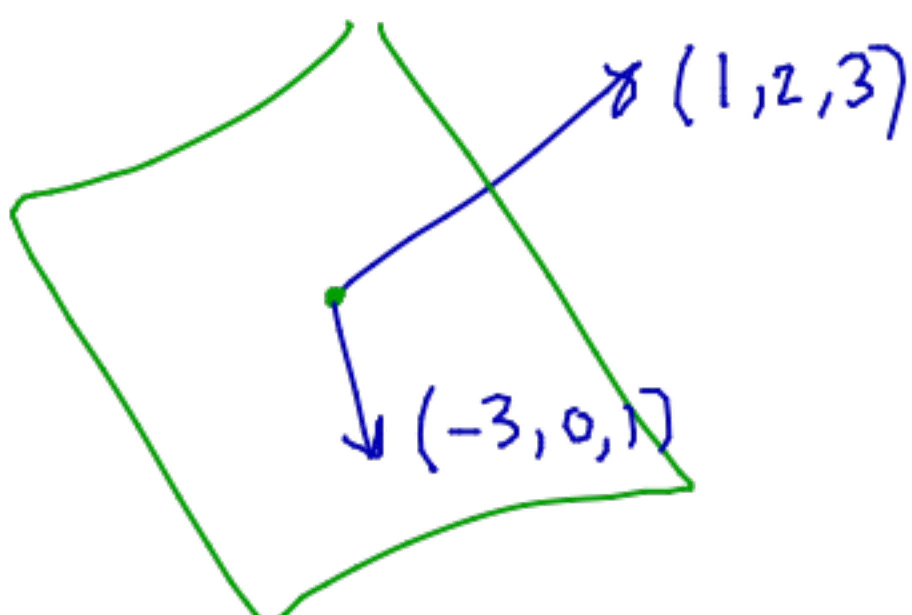
$$\iff \cancel{u\cdot u} + \cancel{v\cdot v} = \cancel{u\cdot u} - 2u\cdot v + \cancel{v \cdot v}$$

$$\iff u \cdot v = 0.$$



Problem. Find a vector in $\mathbb{R}^3$ orthogonal to $(1, 2, 3)$.

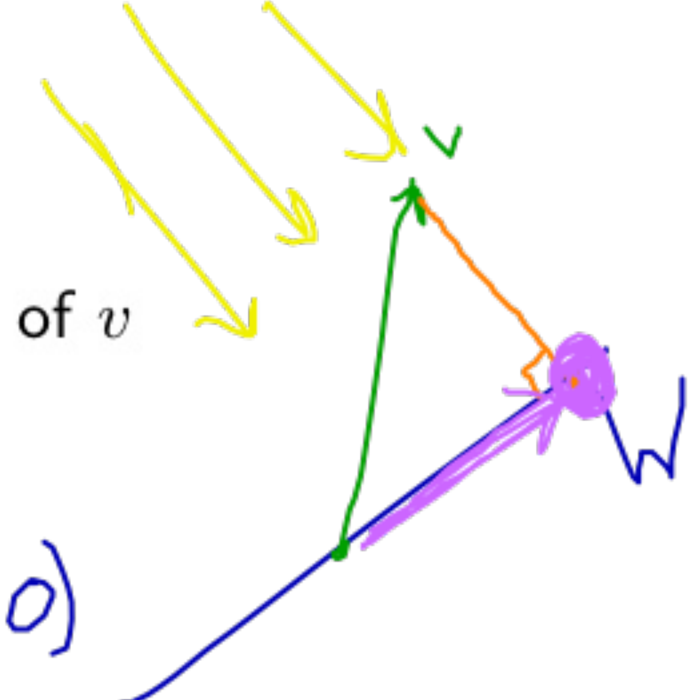
$$(-3,0,1)\cdot(1,2,3) = -3\cdot 1 + 0\cdot 2 + 1\cdot 3 = 0$$



Orthogonal Projections

Let W be a subspace of $\mathbb{R}^n$ and v a vector in $\mathbb{R}^n$.

$\mathrm{proj}_W(v)$ = orthogonal projection to W of v

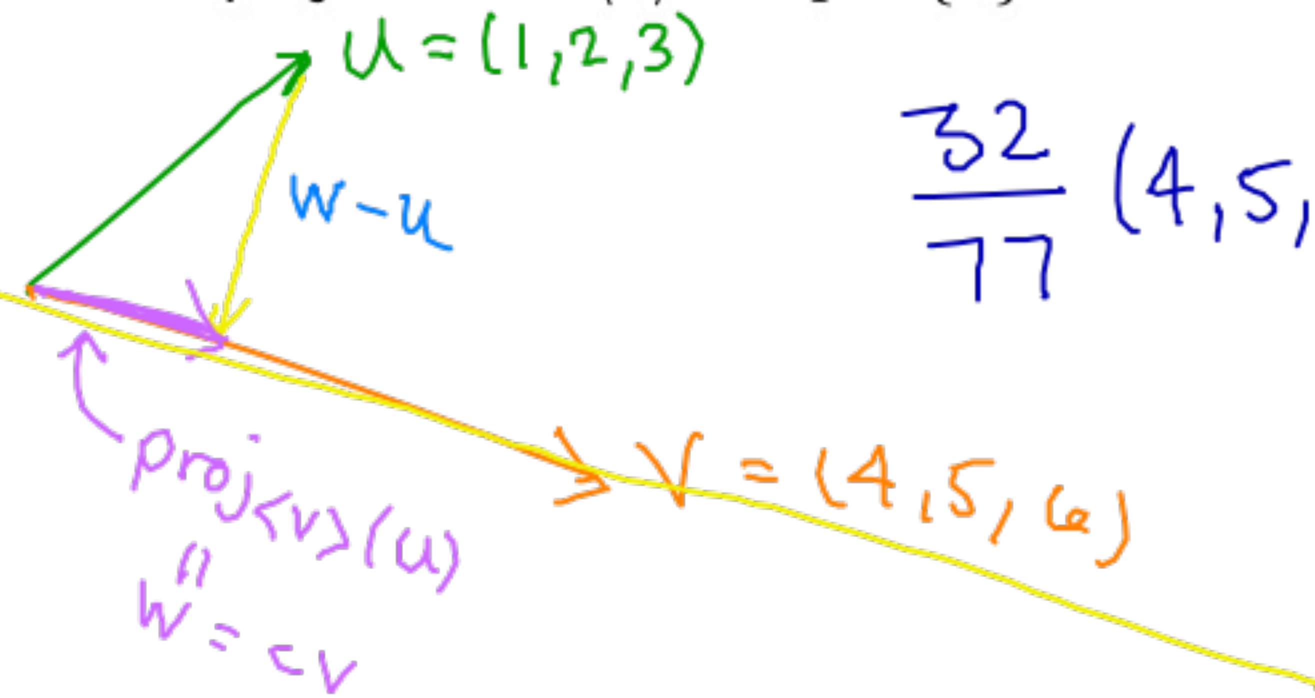


If W is xy-plane

$\mathrm{proj}_W(x,y,z) = (x,y,0)$

If W is x-axis $\mathrm{proj}_W(x,y,z) = (x,0,0)$

Say u and v are vectors in $\mathbb{R}^n$. Can project u to $\langle v \rangle = \mathrm{Span}\{v\}$.



$\frac{32}{77}(4,5,6)$

Fact. $\mathrm{proj}_{\langle v \rangle}(u) = \frac{u \cdot v}{v \cdot v} v$

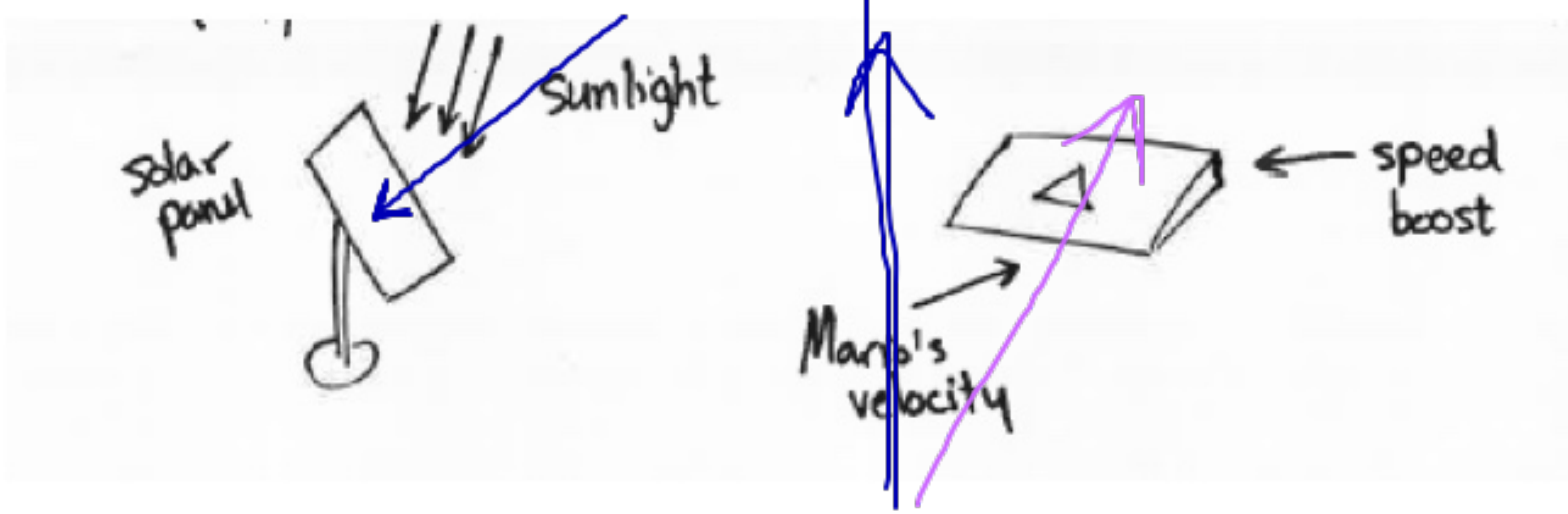
Why?

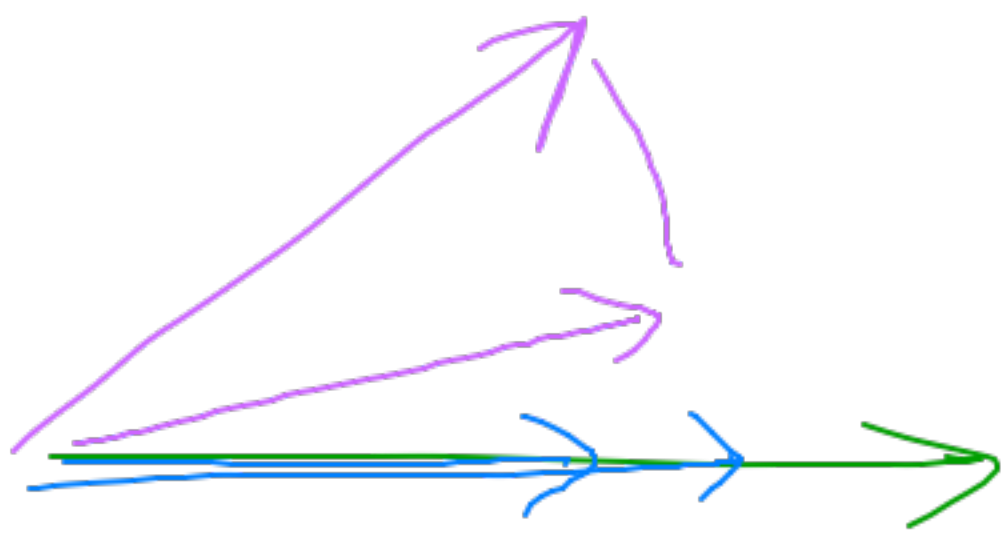
$(cv - u) \cdot v = 0$

$cv \cdot v - u \cdot v = 0 \longrightarrow c = \frac{u \cdot v}{v \cdot v}$

Orthogonal Projections

Many applications, including:



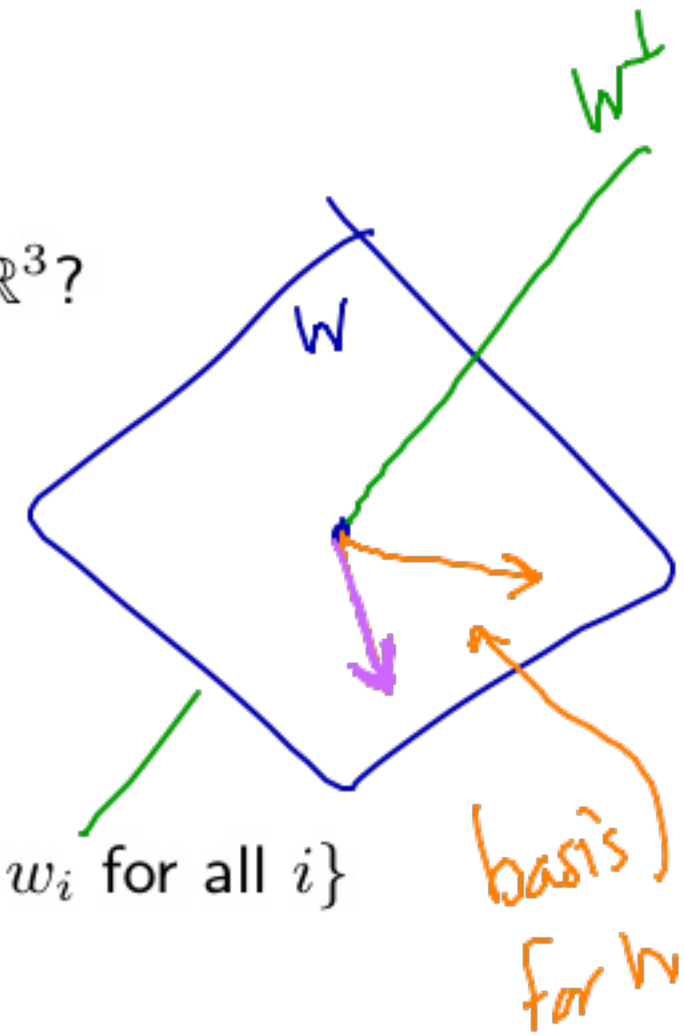


Orthogonal complements

W = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$?

plane.



Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$

Problem. Let $W = \mathrm{Span}\{(1,1,-1)\}$. Find the equation of the plane $W^\perp$.

$$(1,1,-1)\cdot(x,y,z)=0 \qquad x+y-z=0$$

Problem. Let $W = \mathrm{Span}\{(1,1,-1),(-1,2,1)\}$. Find the eqn of the line $W^\perp$.

$$(1,1,-1)\cdot(x,y,z)=0$$
$$(-1,2,1)\cdot(x,y,z)=0$$

$\underline{\text{or}}$ $$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$\boxed{\text{Null space}}$ find param soln.

Orthogonal complements
Finding them

Problem. Let $W = \text{Span}\{(1,1,-1),(-1,2,1)\}$. Find the eqn of the line $W^\perp$.

Theorem. $A = m \times n$ matrix

$$(\text{Row} A)^\perp = \text{Nul}\, A$$

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A