Announcements April 13

- WebWork 6.1 and 6.2 due Thursday
- Quiz on 6.1 and 6.2 on Friday

Written HW 9 due Fri

- Final Exam Wed May 4 8:00-10:50 (Sec H) and Mon May 2 2:50-5:40 (Sec J)
- Office Hours Tue 2-3 and Wed 2-3
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

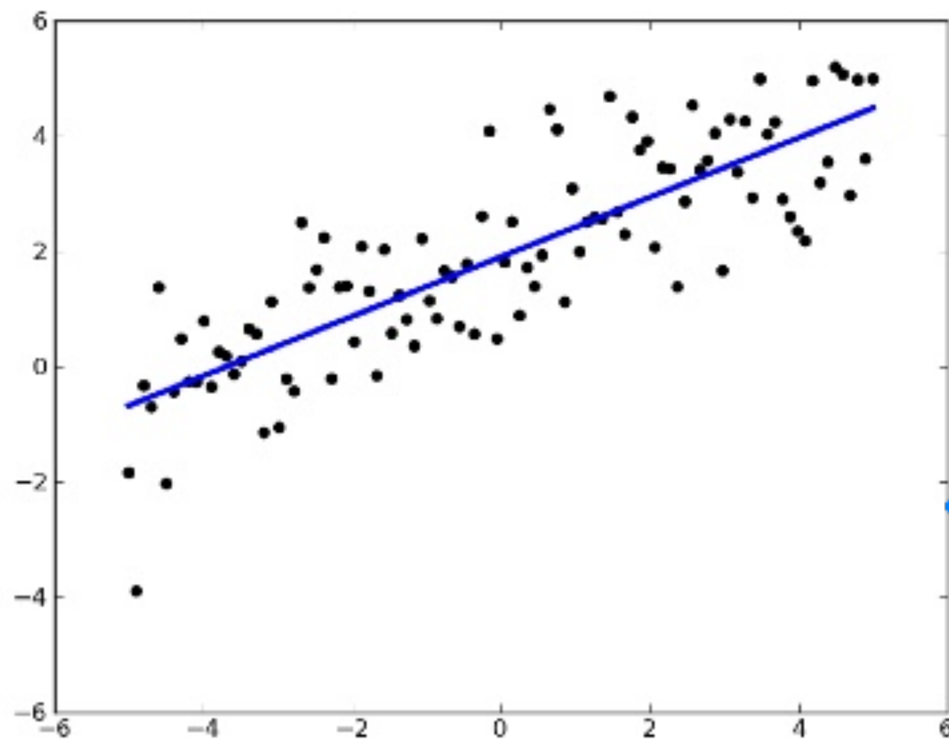
Section 6.2

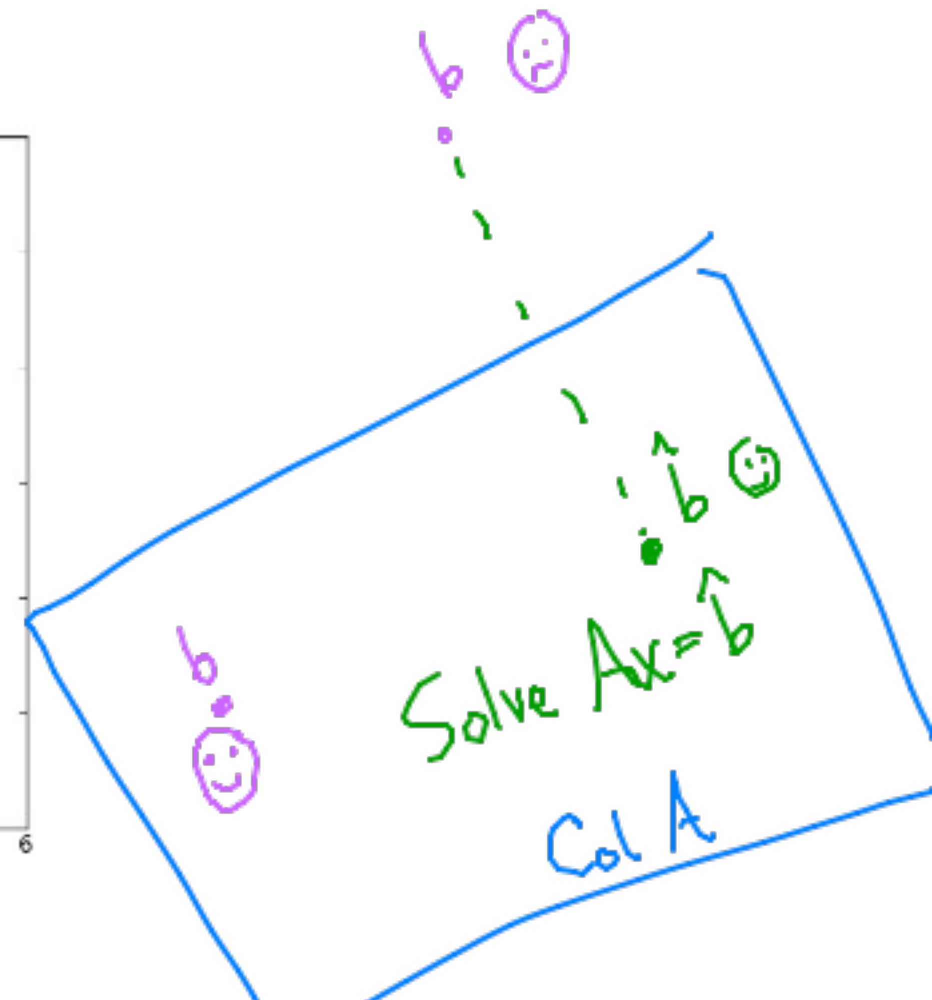
Orthogonal Sets

Where are we?

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



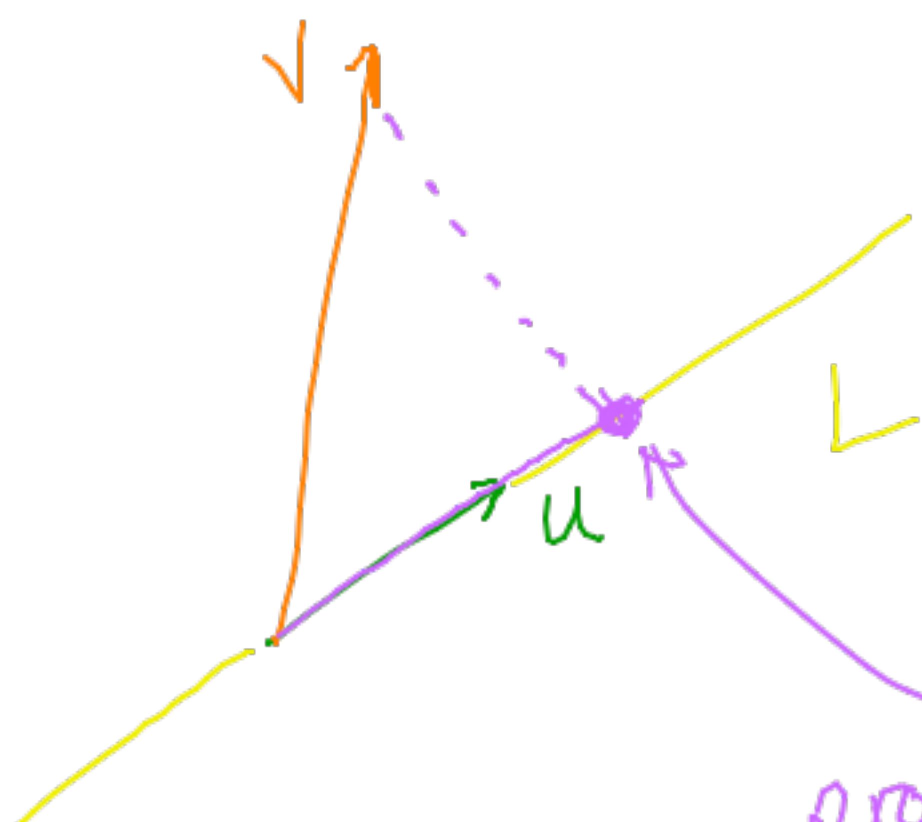


The answer relies on orthogonality. Last time we saw how to project onto a line. Now we will project onto higher-dimensional planes.

Outline

- Orthogonal bases
- A formula for projecting onto any subspace
- Breaking a vector into components

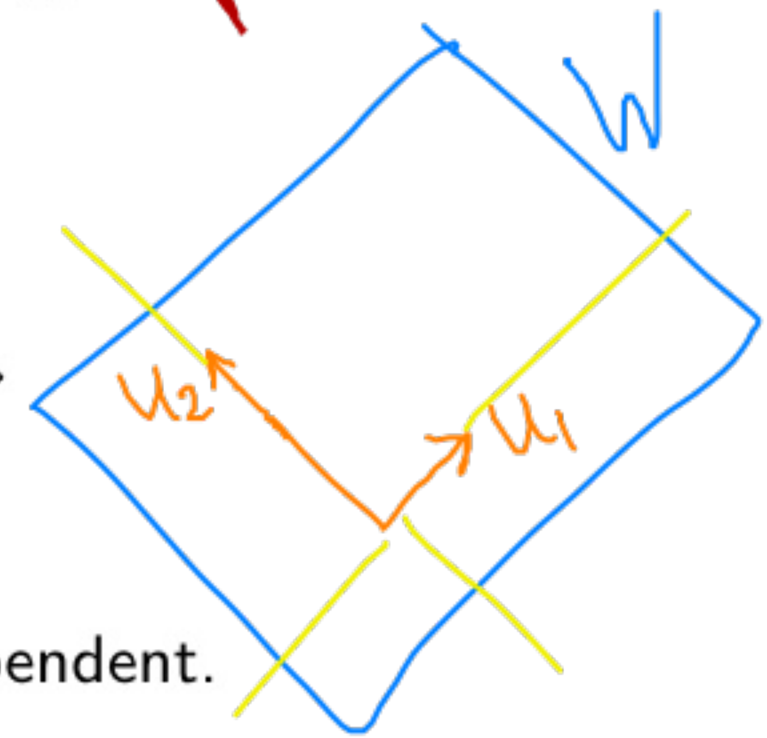
Last time: projected v onto $L = \mathrm{Span}\{u\}$



$$\mathrm{proj}_L(v) = \boxed{\frac{v \cdot u}{u \cdot u}}\, u$$

Orthogonal Sets

A set of vectors is orthogonal if each pair of vectors is orthogonal. It is orthonormal if in addition each vector is a unit vector.



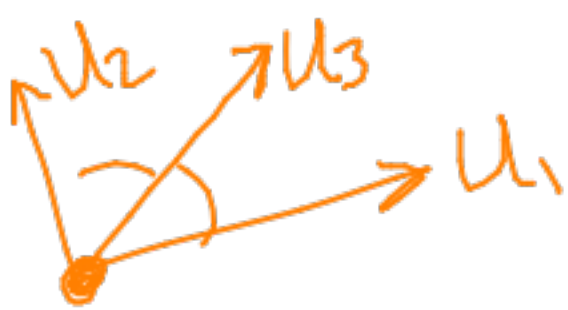
Example.

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Fact. An orthogonal set of nonzero vectors is linearly independent.

Why?

Use vector-by-vector method:



$u_1 \neq 0$ ✓

u_2 not in span of u_1 ✓

etc...

Orthogonal bases

Finding coordinates with respect to orthogonal bases

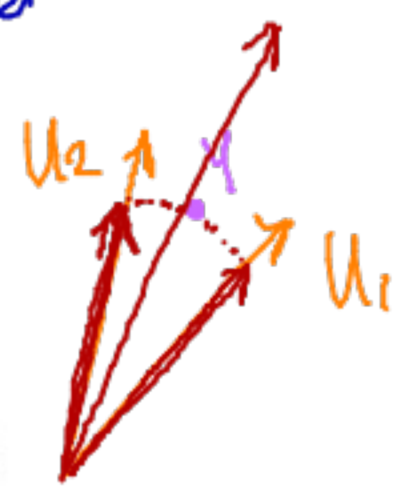
Fact. Say that $\mathcal{B} = \{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and say that y is in W. Then

$$y = \sum_{i=1}^{k} c_i u_i$$

where

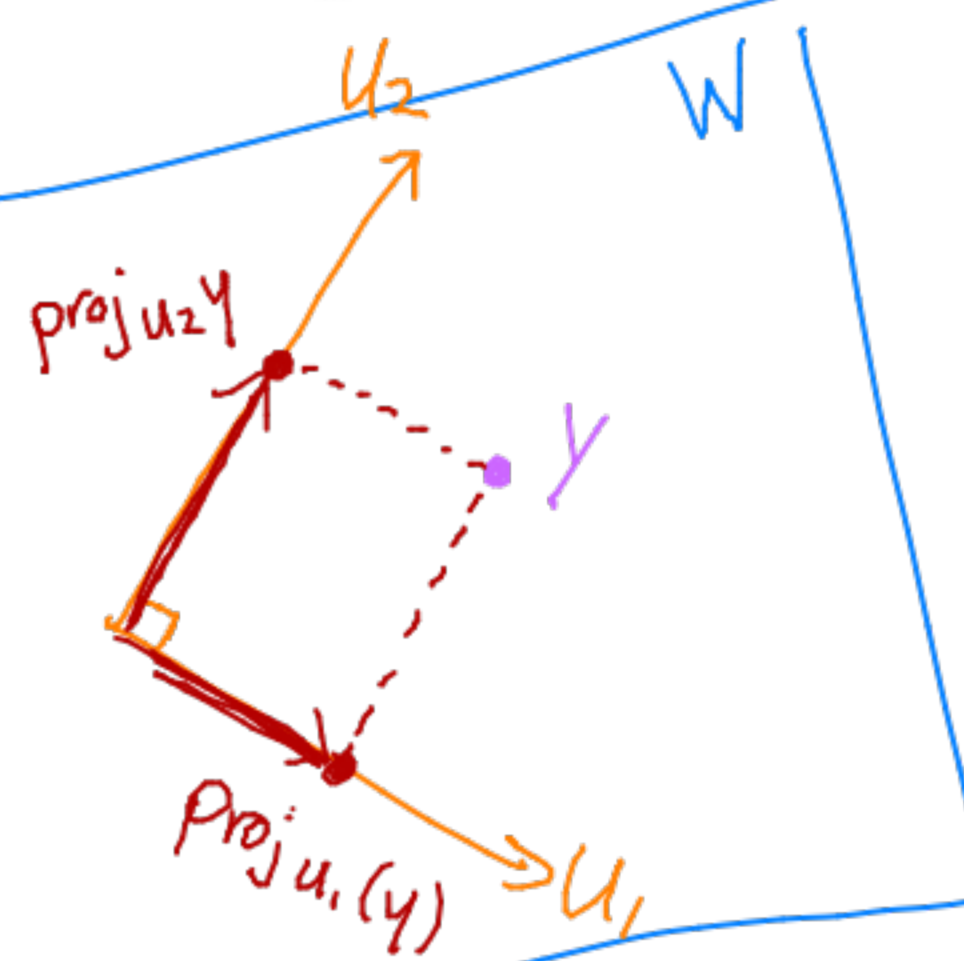
$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

Why we need $u_1, \ldots, u_k$ orthogonal:



In other words:

$$y = \sum_{i=1}^{k} \mathrm{proj}_{\langle u_i \rangle}(y)$$



Why?

$$y = \boxed{\frac{y \cdot u_1}{u_1 \cdot u_1}} u_1 + \boxed{\frac{y \cdot u_2}{u_2 \cdot u_2}} u_2$$

$$y = \underline{\sim 1/2}\, u_1 + \underline{\sim 3/5}\, u_2$$

What happens if y is not in W? The formula still works! But it gives the projection of y to W.

Fact. Say that $\{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and say that y is in W. Then

$$y = \sum_{i=1}^{k} c_i u_i$$

where

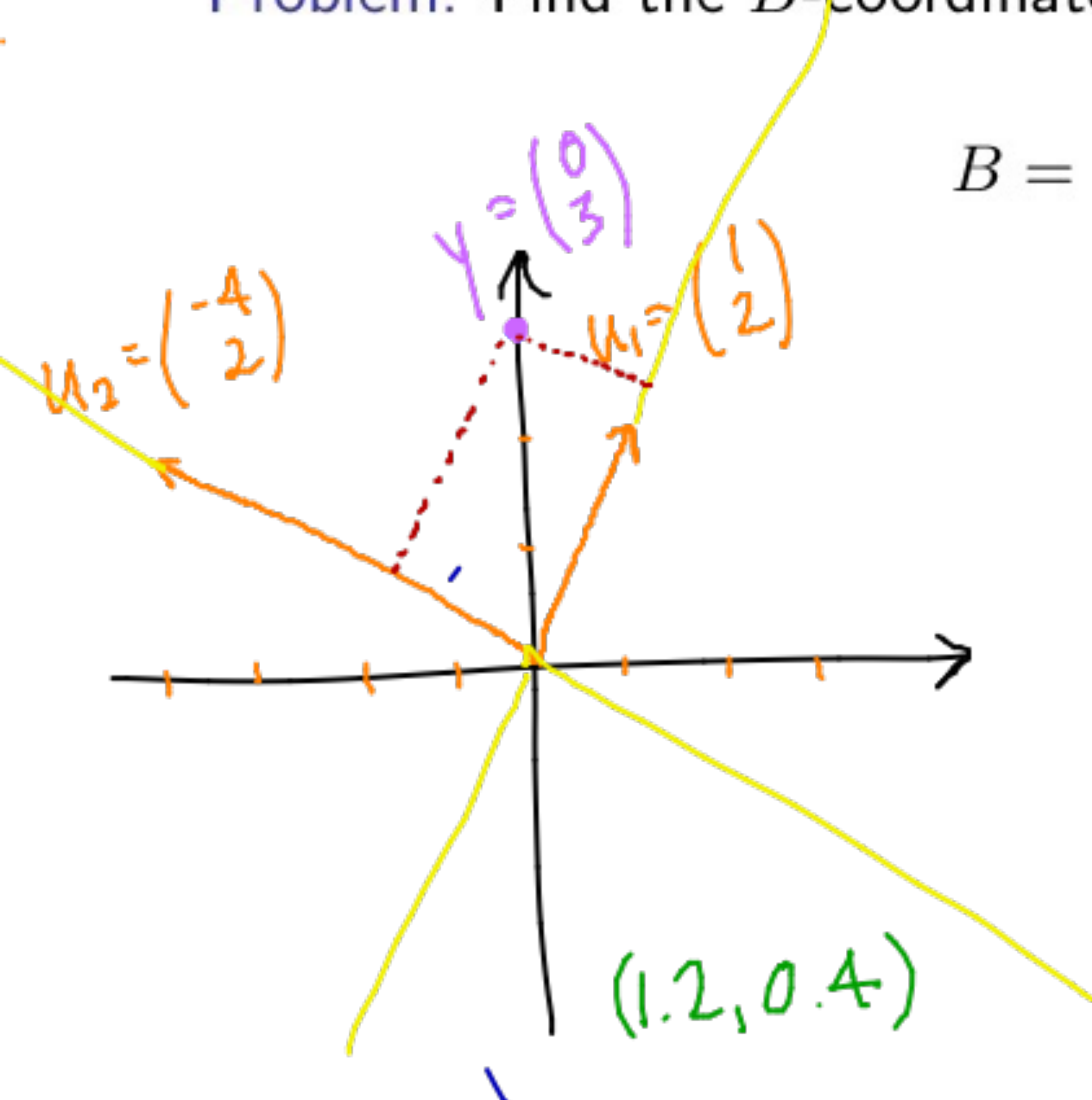
$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

Old way

$$\left(\begin{array}{cc|c} 1 & -4 & 0 \\ 2 & 2 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 6/5 \\ 0 & 1 & 6/20 \end{array}\right)$$

Problem. Find the B-coordinates of $(0,3)$ where

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix} \right\}$$



$$y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \frac{6}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{6}{20}\begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

Answer: $\left(\frac{6}{5}, \frac{6}{20}\right)$

$$\boxed{\text{Have: } y = \frac{6}{5}u_1 + \frac{6}{20}u_2}$$

Fact. Say that $\{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and say that y is in W. Then

$$y = \sum_{i=1}^{k} c_i w_i$$

where

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

$y \cdot u_2$

$(6,1,-8)\cdot(1,-2,1)$

$= 6\cdot 1 + 1(-2) + (-8)1$

$= 6 - 2 - 8$

$= -4$

Problem. Find the B-coordinates of $\underset{y\,=}{(6,1,-8)}$ where

$$B = \left\{ \underset{u_1}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{u_2}{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}, \underset{u_3}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \right\}$$

$$\left(\frac{y \cdot u_1}{u_1 \cdot u_1}, \frac{y \cdot u_2}{u_2 \cdot u_2}, \frac{y \cdot u_3}{u_3 \cdot u_3} \right)$$

$$\left(\frac{-1}{3}, \frac{-4}{6}, \frac{14}{2} \right)$$

So: $\begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + -\frac{4}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \frac{14}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

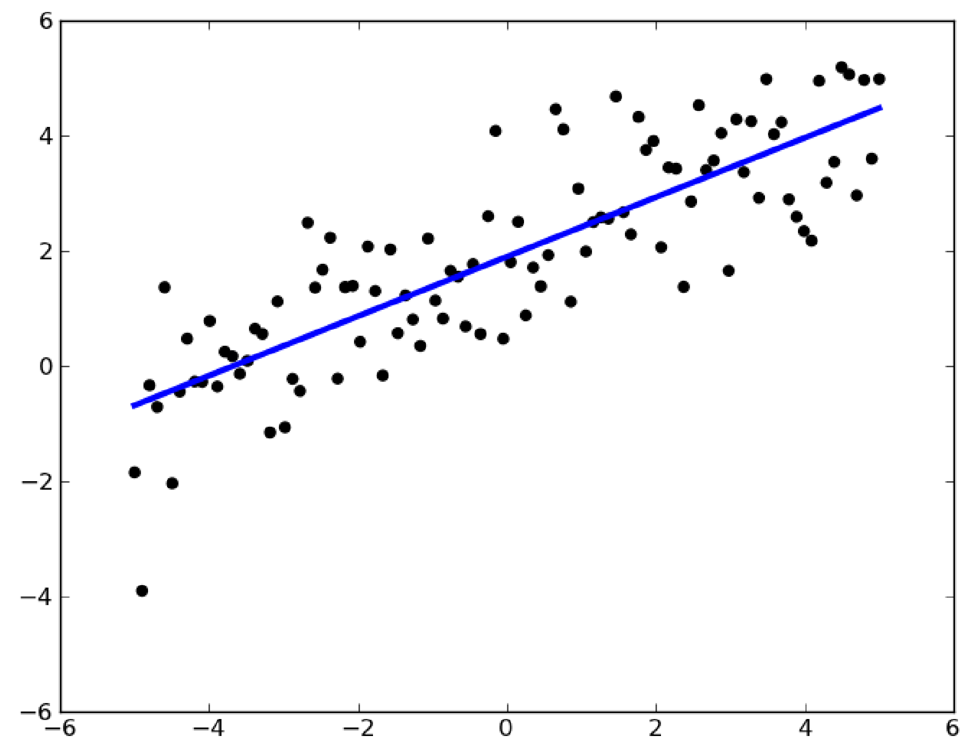
Section 6.3

Orthogonal projections

Where are we?

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



The answer relies on orthogonality.

Outline

- Projecting onto any subspace: a formula
- Projections and best possible solutions

Orthogonal projection

Projecting onto a line

Recall:

$$\mathrm{proj}_{\langle u\rangle}(v) = \frac{v\cdot u}{u\cdot u}u$$

Can use this to break v into two components:

$$v = v_L + v_{L^\perp}$$

where $L = \langle u\rangle$ and v_L is $\mathrm{proj}_{\langle u\rangle}(v)$ and $v_{L^\perp} = v - v_L$.

Problem. Let $u = (1,2)$ and $L = \langle u\rangle$. Let $v = (1,1)$. Write v as $v_L + v_{L^\perp}$.

Next: replace L with any subspace.

Orthogonal projection

Projecting onto any subspace

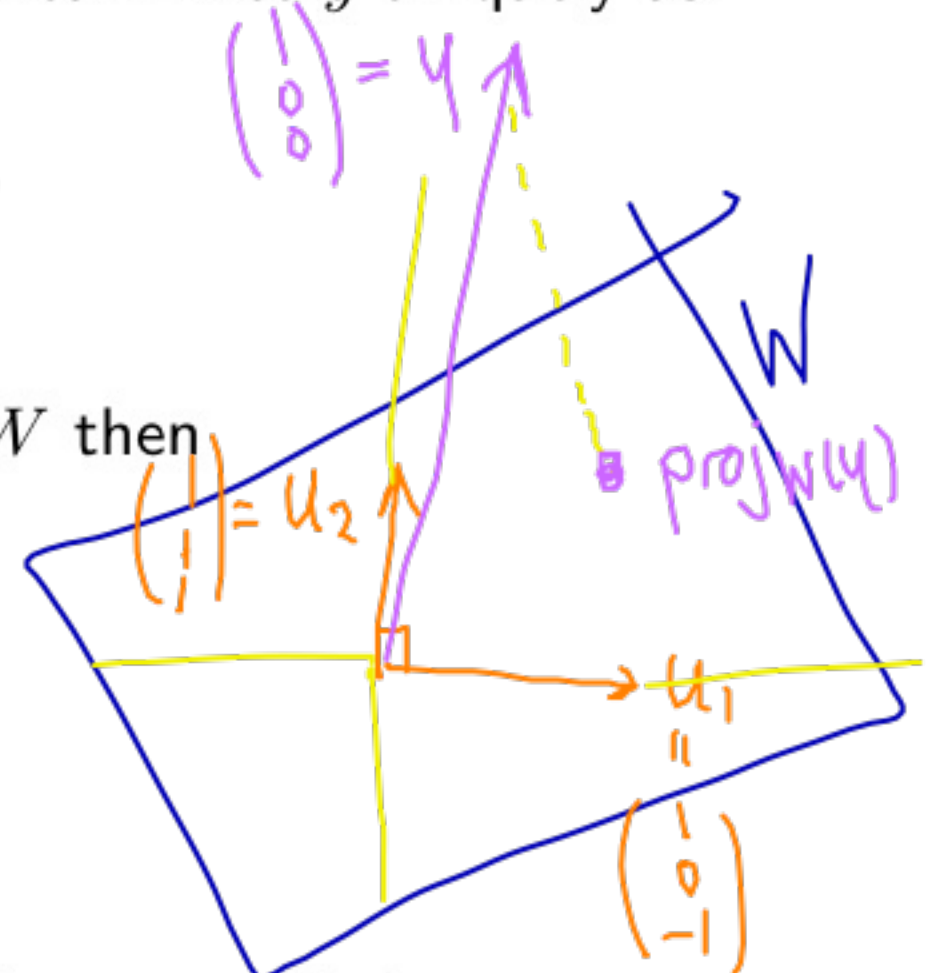
$$(57, 32, 99) \cdot (1, 0, 0) = 57$$

Theorem. Say W a subspace of $\mathbb{R}^n$ and y in $\mathbb{R}^n$. We can write y uniquely as:

$$y = y_W + y_{W^\perp}$$

with y_W in W and $y_{W^\perp}$ in $W^\perp$.

Moreover, if $\{u_1, \ldots, u_k\}$ is an orthogonal basis for W then



$$\mathrm{proj}_W(y) = y_W = \sum_{i=1}^{k} \frac{y \cdot u_i}{u_i \cdot u_i} u_i$$

This y_W is $\mathrm{proj}_W(y)$.

Problem. Let $W = \mathrm{Span}\left\{ \begin{array}{cc} u_1 & u_2 \\ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, & \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{array} \right\}$ and $y = e_1$. Find y_W.

$$\mathrm{proj}_W(y) = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}5/6\\1/3\\-1/6\end{pmatrix}$$

Find A so that T_A is orthogonal projection onto

$$W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$A = \left(\begin{array}{c|c|c} \mathrm{proj}_W(e_1) & \mathrm{proj}_W(e_2) & \mathrm{proj}_W(e_3) \end{array} \right)$$

$$= \left(\begin{array}{c|c|c} 5/6 & \text{do prev page with } e_2 \text{ instead of } e_1 & \text{now with } e_3 \\ 1/3 & & \\ -1/6 & & \end{array} \right)$$

Orthogonal projection

$T_X \circ T_Y = T_{XY}$

Poll

Suppose T_A is orthogonal projection onto a plane in $\mathbb{R}^3$. What is A^2 equal to?

1. A
2. A^{-1}
3. $-A$
4. 0
5. I_n
6. $-I_n$

$$\boxed{T_A \circ T_A} = T_{A^2}$$
$$\boxed{T_A \circ T_A} = T_A$$

4

Hint: What $T_A \circ T_A$? Ans T_A

$T_A \circ T_A(v) = T_A(T_A(v))$

Best approximation

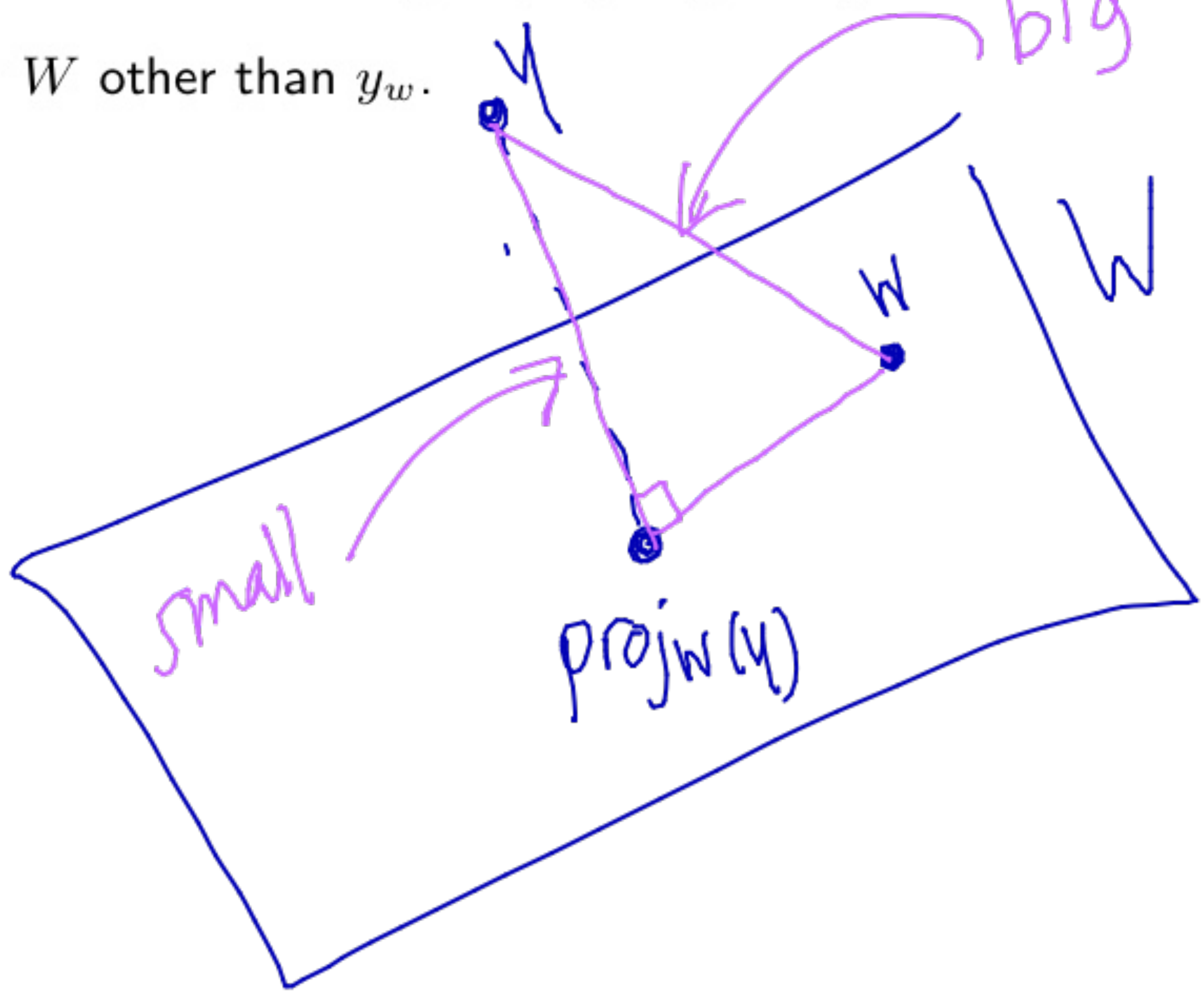
W = subspace of $\mathbb{R}^n$

Fact. The projection y_W is the point in W closest to y. In other words:

$$\|y - y_w\| < \|y - w\|$$

for any w in W other than y_w.

Why?



Best approximation

Problem. Find the distance from e_1 to $W = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$.