Announcements April 18

- CIOS open: additional dropped quiz for 85% response rate
- WebWork 6.3 and 6.4 due Thursday

  Written HW Friday
- Quiz on 6.3 and 6.4 on Friday
- WebWork 6.5 due Sunday (not graded)
- Review on Monday in class; post questions on Piazza using final_exam tag
- Final Exam [redacted] and Mon May 2 2:50-5:40 (Sec J)
- Office Hours Tue 2-3 and Wed 2-3
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 6.4

The Gram–Schmidt Process

Where are we?

only works if $v_1 \perp v_2$: $\text{proj}_{\text{Col}A}(b) = \text{proj}_{\langle v_1 \rangle}(b) + \text{proj}_{\langle v_2 \rangle}(b)$

We have one more main goal.

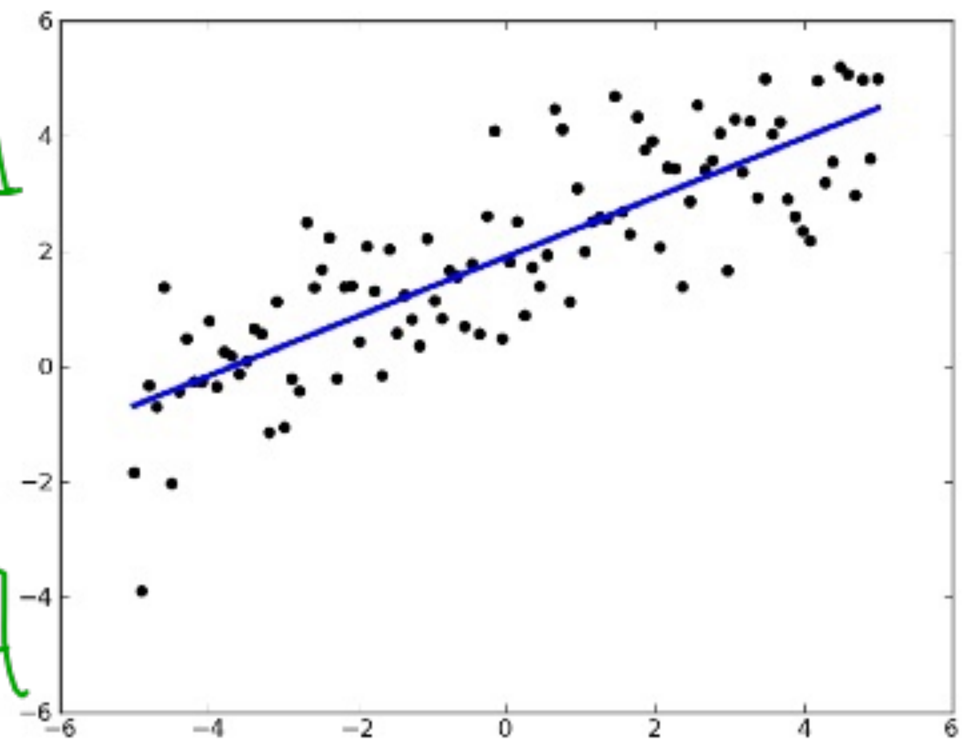
What if we can't solve $Ax = b$? How can we solve it as closely as possible?

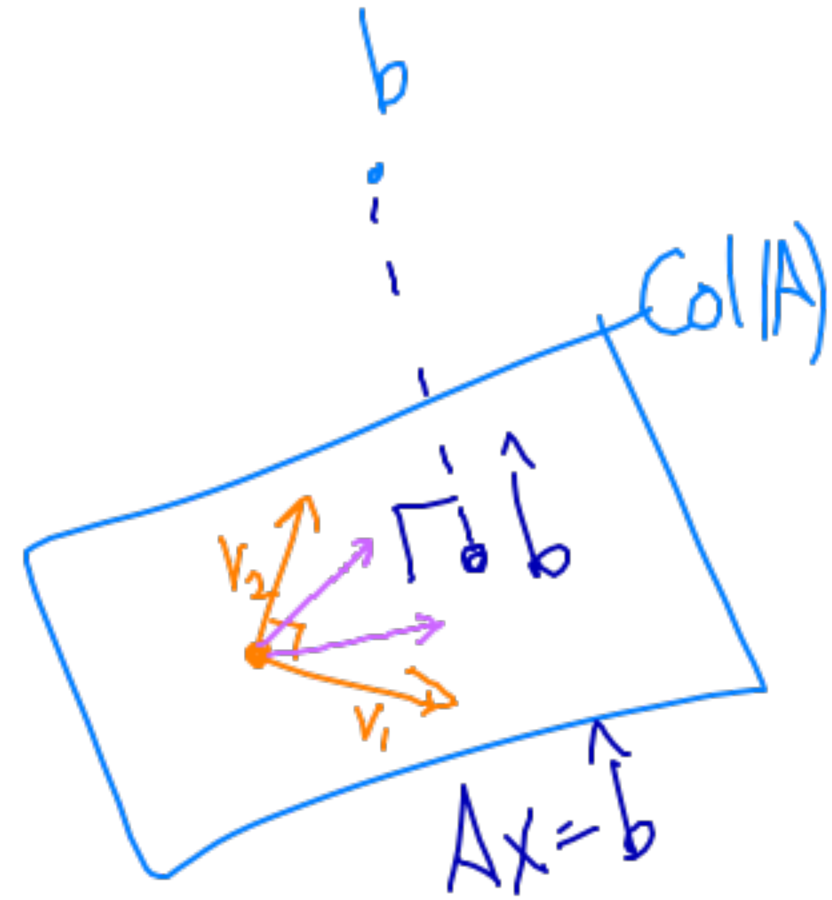
$W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$

Q. Find basis for $W^\perp$

A. $\text{Nul} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Why? $(\text{Row}A)^\perp = \text{Nul}A$





To solve $Ax = b$ as closely as possible, we orthogonally project b onto $\text{Col}(A)$. We know how to do this if we have an orthogonal basis. But what if we don't?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{matrix} (1,2,3) \cdot (x,y,z) = 0 \\ (4,5,6) \cdot (x,y,z) = 0 \end{matrix}$$

Outline

- The Gram–Schmidt process: turn any basis into an orthogonal one
- QR factorization
- Application to eigenvalue computations

Gram–Schmidt Process

With two vectors

Find an orthogonal basis for $W = \mathrm{Span}\{u_1, u_2\}$, where

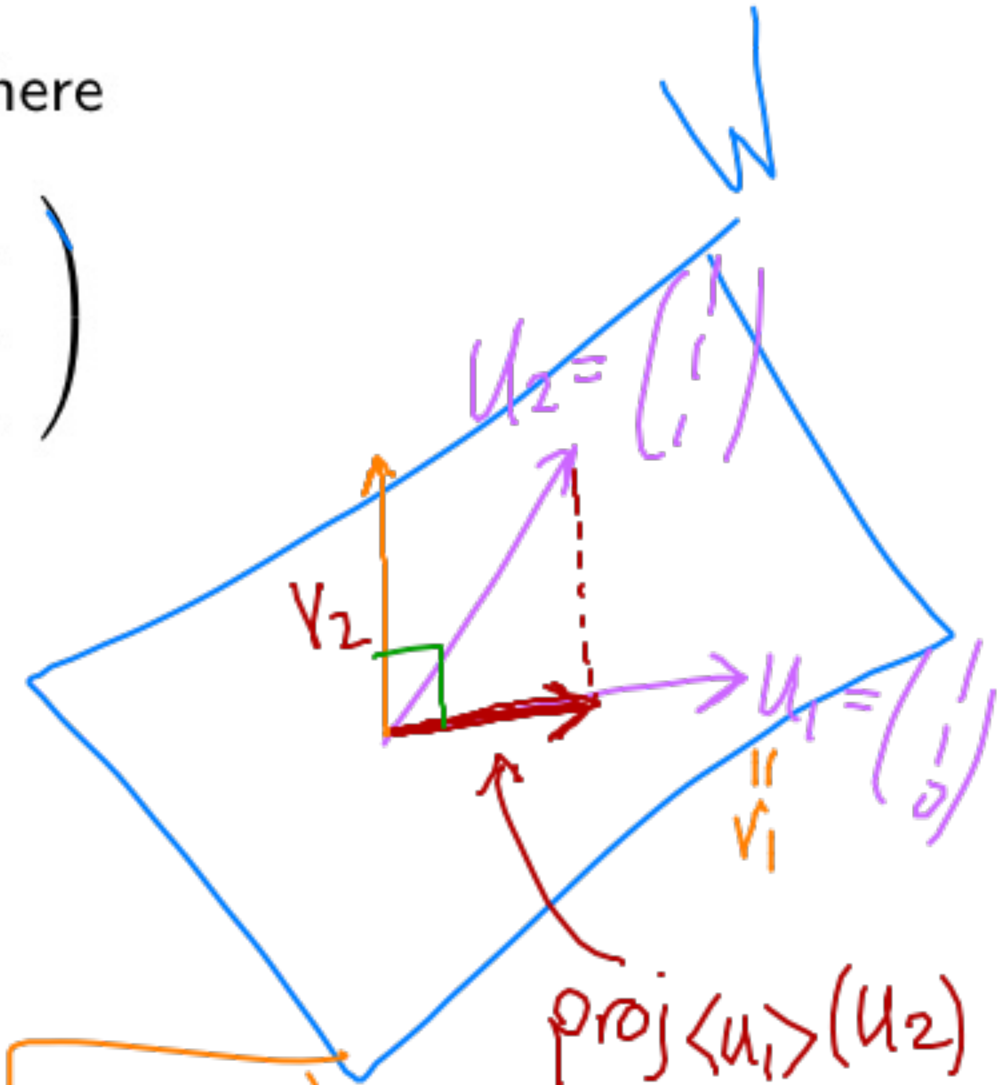
$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$V_1 = U_1 = \boxed{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

$$V_2 = U_2 - \mathrm{proj}_{\langle \overset{v_1}{\cancel{u_1}} \rangle}(U_2)$$

$$= U_2 - \frac{v_1 \cdot u_2}{v_1 \cdot v_1} v_1$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$



Check $v_1 \cdot v_2 = 0$

$v_1 \cdot v_2 =$

$v_1 \cdot \left(u_2 - \frac{v_1 \cdot u_2}{v_1 \cdot v_1} v_1\right)$

$= v_1 \cdot u_2 - \frac{v_1 \cdot u_2}{v_1 \cdot v_1} v_1 \cdot v_1$

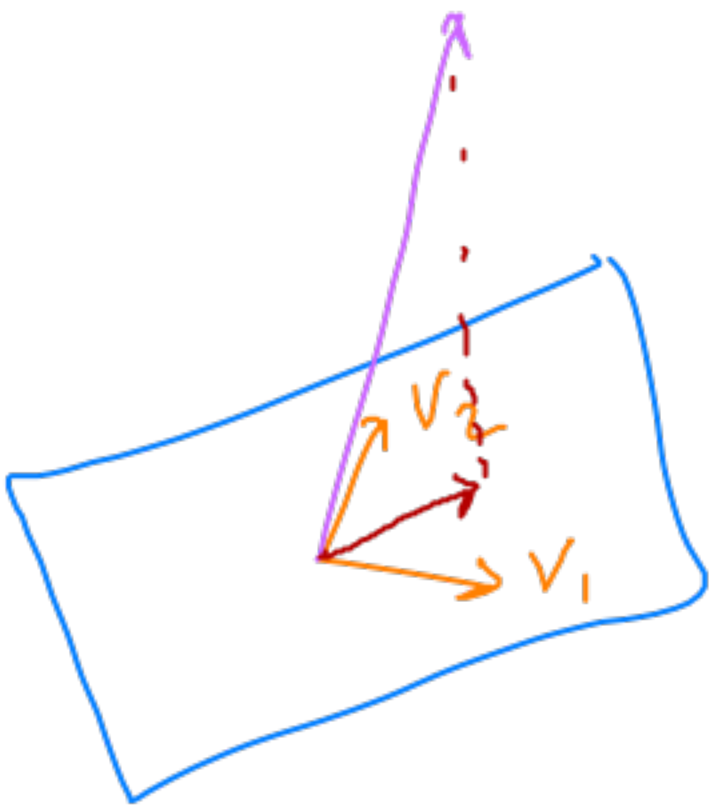
$= 0$

Gram–Schmidt Process

With three vectors

Find an orthogonal basis for $W = \mathrm{Span}\{u_1, u_2, u_3\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$



$$v_1 = u_1 = \boxed{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

$$v_2 = u_2 - \mathrm{proj}_{\langle v_1 \rangle}(u_2) = \boxed{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

Span$\{v_1\}$

$$v_3 = u_3 - \mathrm{proj}_{\mathrm{Span}\{v_1, v_2\}}(u_3)$$

$$= u_3 - \left(\mathrm{proj}_{\langle v_1 \rangle}(u_3) + \mathrm{proj}_{\langle v_2 \rangle}(u_3)\right)$$

$$= u_3 - \frac{v_1 \cdot u_3}{v_1 \cdot v_1} v_1 - \frac{v_2 \cdot u_3}{v_2 \cdot v_2} v_2$$

$$= \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \frac{4}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$$

Gram-Schmidt Process

Example

Theorem. Say $\{u_1, \ldots, u_k\}$ is a basis for a nonzero subspace of $\mathbb{R}^n$. Define:

$$
\begin{aligned}
v_1 &= u_1 \\
v_2 &= u_2 - \mathrm{proj}_{\mathrm{Span}\{v_1\}}(u_2) \\
v_3 &= u_3 - \mathrm{proj}_{\mathrm{Span}\{v_1, v_2\}}(u_3) \\
&\vdots \\
v_k &= u_k - \mathrm{proj}_{\mathrm{Span}\{v_1, \ldots, v_{k-1}\}}(u_k)
\end{aligned}
$$

Then $\{v_1, \ldots, v_k\}$ is an orthogonal basis for $\mathrm{Span}\{u_1, \ldots, u_k\}$.

Gram–Schmidt Process

With three vectors

Find an orthogonal basis for $W = \mathrm{Span}\{u_1, u_2, u_3\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix}$$

$$v_1 = u_1 = \boxed{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$$

$$-\frac{2}{5} = -\frac{10}{25}$$

$$v_2 = u_2 - \mathrm{proj}_{\langle v_1 \rangle}(u_2) = u_2 - \frac{v_1 \cdot u_2}{v_1 \cdot v_1} v_1$$

$$= \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} - \frac{6}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{pmatrix}}$$

$$1 = \frac{4}{4}$$

$$v_3 = u_3 - \mathrm{proj}_{\mathrm{span}\{v_1, v_2\}}(u_3)$$

$$= u_3 - \mathrm{proj}_{\langle v_1 \rangle}(u_3) - \mathrm{proj}_{\langle v_2 \rangle}(u_3)$$

$$= u_3 - \frac{v_1 \cdot u_3}{v_1 \cdot v_1} v_1 - \frac{v_2 \cdot u_3}{v_2 \cdot v_2} v_2 \quad \boxed{= \cdots (2, -2, 2, -2)}$$

QR Factorization

Theorem. Say A is a [redacted] matrix with linearly independent columns. Then

$$A = QR$$

where Q has orthonormal columns and R is upper triangular with positive diagonal entries.

$QQ^T = I \rightsquigarrow Q^{-1} = Q^T$

Columns of Q are the vectors obtained from Gram–Schmidt, with normalized columns.

The entries of R come from the steps in the Gram–Schmidt process, with normalized rows. In our first 3×3 example:

$$\hat{Q} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \hat{R} = \begin{pmatrix} 1 & \boxed{1} & \boxed{2} \\ 0 & 1 & \boxed{1} \\ 0 & 0 & 1 \end{pmatrix}$$

$\hat{Q}\hat{R} = A$

The first $\boxed{1}$ comes from: $v_2 = u_2 - 1 \cdot v_1$

The other $\boxed{2}$ and $\boxed{1}$ come from $v_3 = u_3 - 2 \cdot v_1 - 1 \cdot v_2$

QR Factorization

Theorem. Say A is an $n \times n$ matrix with linearly independent columns. Then

$$A = QR$$

where Q has orthonormal columns and R is upper triangular with positive diagonal entries.

In our first 3×3 example:

$$\hat{Q} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \hat{R} = \begin{pmatrix} 1 & \boxed{1} & \boxed{2} \\ 0 & 1 & \boxed{1} \\ 0 & 0 & 1 \end{pmatrix}$$

result of G-S

Steps of G-S

To find Q and R, scale columns of $\hat{Q}$ to make them unit vectors and scale the corresponding rows of $\hat{R}$ by the inverse.

$$\underbrace{\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{2} \end{pmatrix}}_{R} = A$$

QR Factorization

Example

$$\|(1,1,1,1)\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$$

Find the QR factorization of

div. by 2 div by 5

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\hat{Q} = \begin{pmatrix} 1 & -5/2 & 2 \\ 1 & 5/2 & -2 \\ 1 & 5/2 & 2 \\ 1 & -5/2 & -2 \end{pmatrix} \qquad \hat{R} = \begin{pmatrix} 1 & \boxed{3/2} & \boxed{1} \\ 0 & 1 & \boxed{-2/5} \\ 0 & 0 & 1 \end{pmatrix}$$

mult by 2

mult by 5

$$Q = \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{pmatrix} \qquad R = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

<u>or</u> If you find Q

$$R = Q^{-1}A = Q^TA$$

<u>Check</u> $A = QR$.

QR Factorization

What is it used for?

Say A is an $n \times n$ matrix.

$R_1(Q_1R_1)R_1^{-1}$

Do:

$$\begin{aligned}
&\boxed{A = Q_1R_1} && \text{QR factorization} \\
A_1 &= \boxed{R_1Q_1} && \text{swap Q and R} \\
&= \boxed{Q_2R_2} && \text{and find the QR factorization of the result} \\
A_2 &= \boxed{R_2Q_2} && \text{swap Q and R} \\
&\vdots
\end{aligned}$$

The A_k converge to an upper triangular matrix and the diagonal entries (quickly!) converge to the eigenvalues.