Announcements April 20

- CIOS open: additional dropped quiz for 85% response rate
- WebWork 6.3 and 6.4 due Thursday
- Written Homework 10 due Friday
- Quiz on 6.3 and 6.4 on Friday
- WebWork 6.5 due Sunday (not graded)
- Review on Monday in class; post questions on Piazza using final_exam tag
- Final Exam [redacted] Mon May 2 2:50-5:40 (Sec J)
- Office Hours Tue 2-3 and Wed 2-3
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
    - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
    - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
    - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

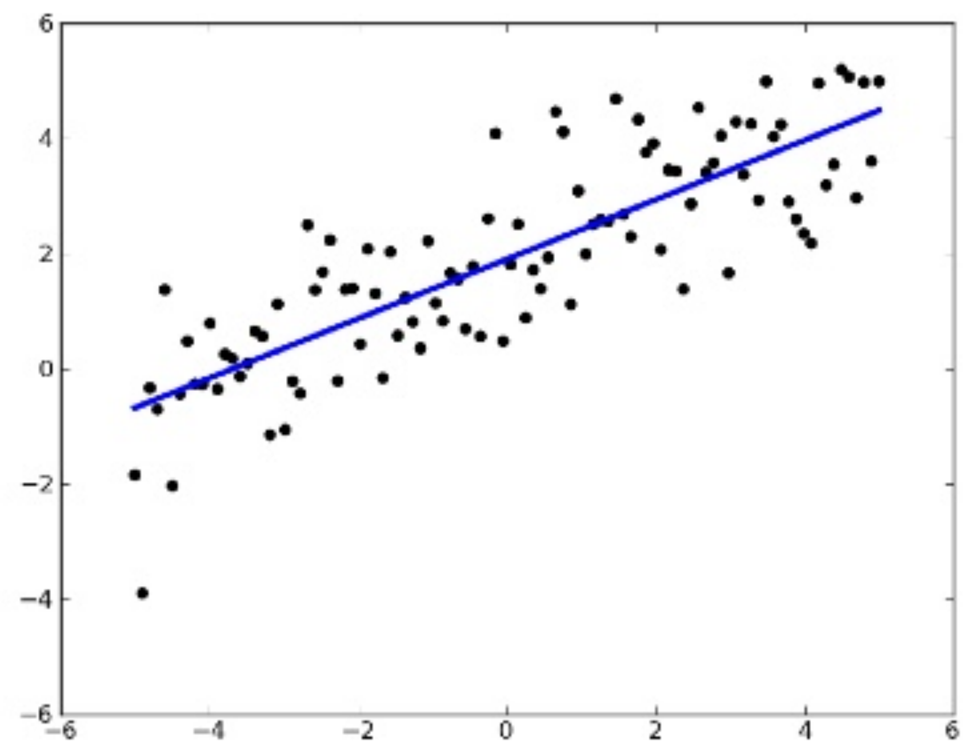
Section 6.5

Least Squares Problems

Where are we?

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



To solve $Ax = b$ as closely as possible, we orthogonally project b onto $\mathrm{Col}(A)$; call the result $\hat{b}$. Then solve $Ax = \hat{b}$. This is the *least squares solution* to $Ax = b$.

Outline

- The method of least squares
- Application to best fit lines/planes
- Application to best fit curves

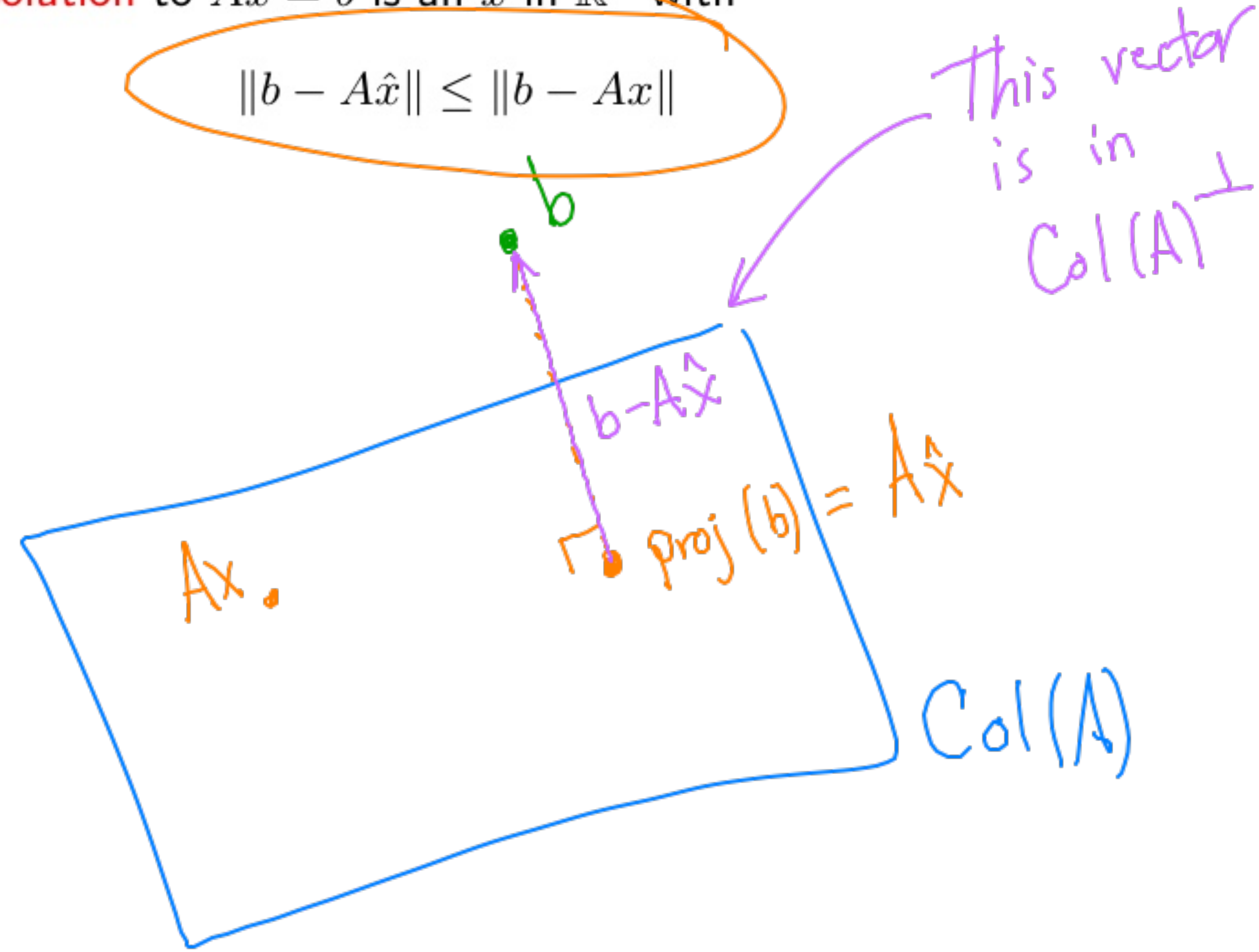
Least squares solutions

$A = m \times n$ matrix.

A least squares solution to $Ax = b$ is an $\hat{x}$ in $\mathbb{R}^n$ with

$$\|b - A\hat{x}\| \leq \|b - Ax\|$$

for all x in $\mathbb{R}^n$



Least squares solutions

A least squares solution to $Ax = b$ is an $\hat{x}$ in $\mathbb{R}^n$ with

$$\|b - A\hat{x}\| \leq \|b - Ax\|$$

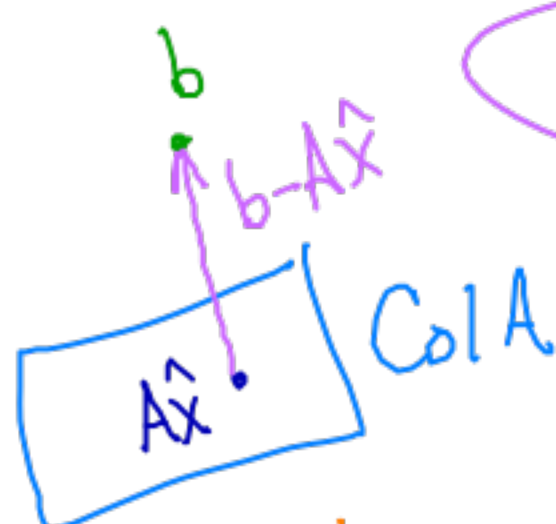
for all x in $\mathbb{R}^n$

Theorem. The least squares solutions to $Ax = b$ are the solutions to

$$(A^T A)x = (A^T b)$$

$\longleftarrow$ This is an $Ax=b$ problem.

Why?



$b - A\hat{x}$ in $(\text{Col} A)^\perp$

$\Longleftrightarrow A^T(b - A\hat{x}) = 0$

$\Longleftrightarrow A^T b - A^T A \hat{x} = 0$

Another method

① Use GS to find $\perp$ basis for Col A

② Use our formula & ① to project b

③ Solve $Ax = \text{proj}(b)$

Least squares solutions

Example

How good of a solution did we find? $\left\| \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\| = \sqrt{6}$

Find the least squares solutions to $Ax = b$ for the following A and b:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

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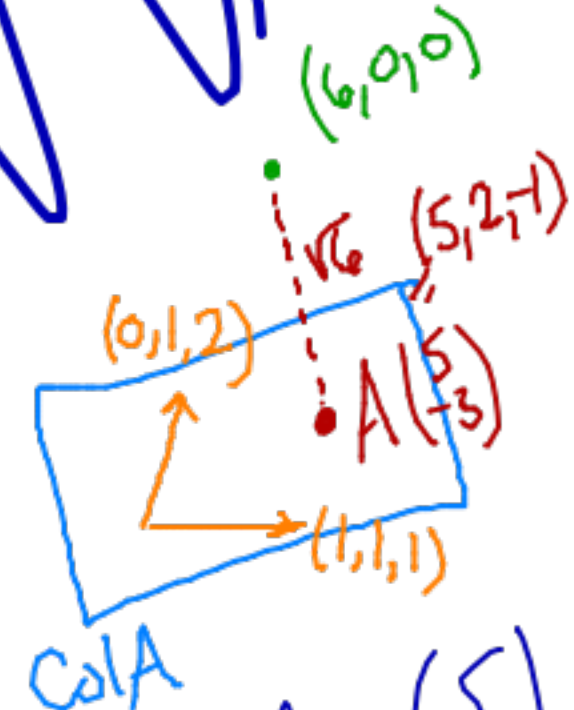


$$A^TA = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$$\hat{x} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

Solve $\left(\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 5 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 3 & 3 & 6 \\ 0 & 2 & -6 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -3 \end{array}\right) \rightsquigarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \end{pmatrix}$

Least squares solutions

Example

Find the least squares solutions to $Ax = b$ for the following A and b:

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Least squares solutions

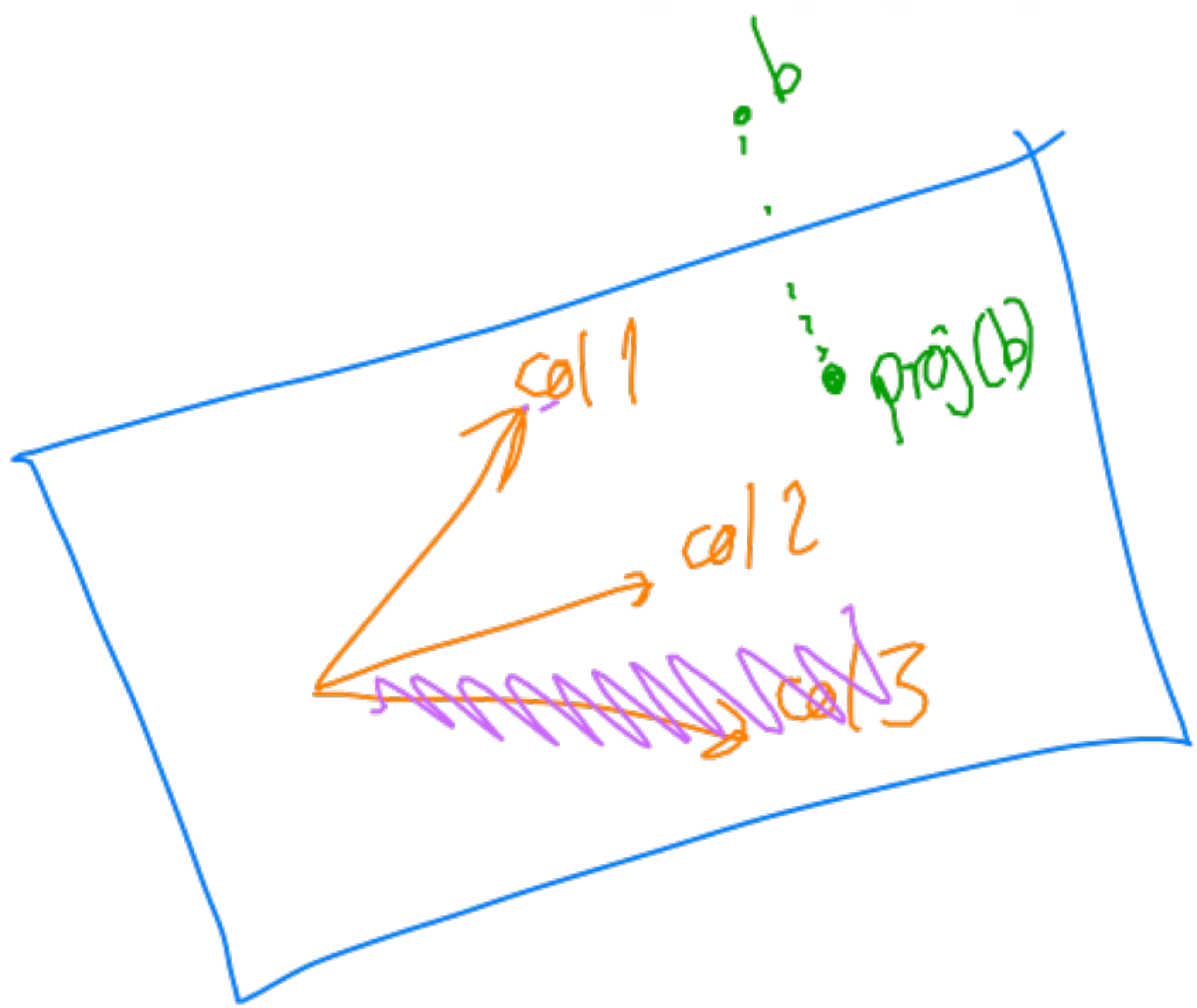
Theorem. Let A be an $m \times n$ matrix. The following are equivalent:

1. $Ax = b$ has a unique least squares solution for all b in $\mathbb{R}^n$
2. The columns of A are linearly independent
3. $A^T A$ is invertible

In this case the least squares solution is $(A^T A)^{-1}(A^T b)$.

$A^T A x = A^T b$

unique soln $\Longleftrightarrow$ $A^T A$ inv. (IMT)



Col A

Application

Best fit lines

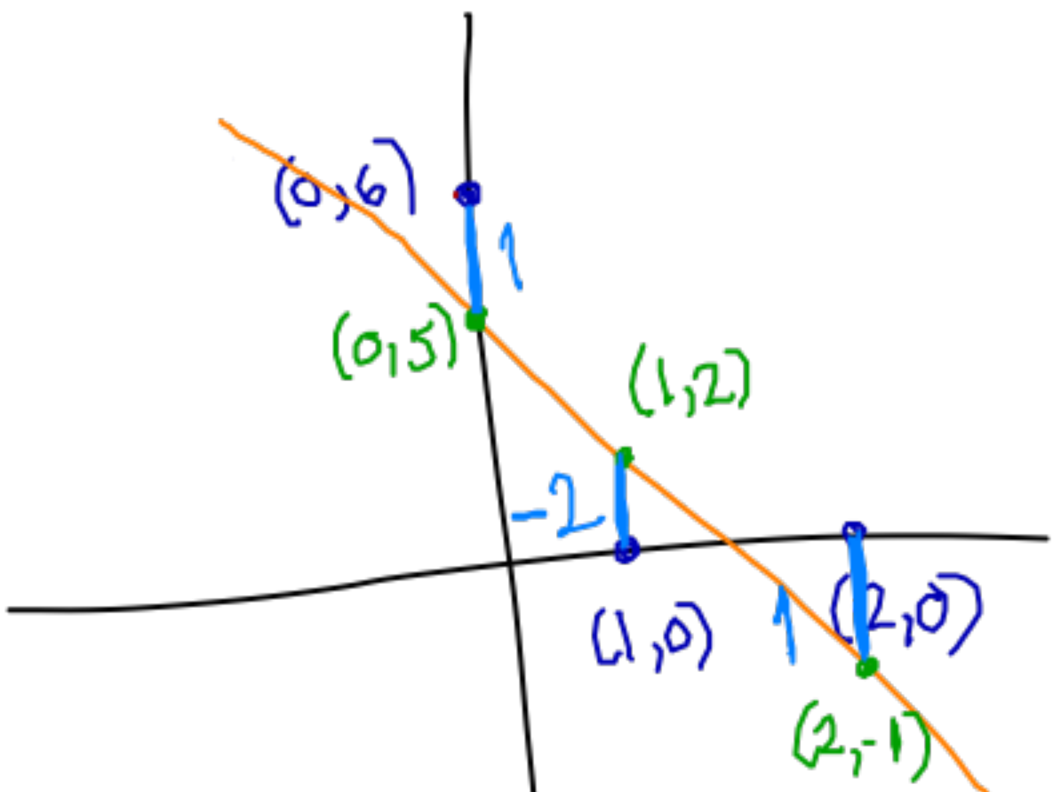
Problem. Find the best-fit line through $(0,6)$, $(1,0)$, and $(2,0)$.

General eqn of a line:

$$C + Dx = y$$

$$\rightsquigarrow \quad \begin{aligned} C + D\cdot 0 &= 6 \\ C + D\cdot 1 &= 0 \\ C + D\cdot 2 &= 0 \end{aligned}$$

$$\rightsquigarrow \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} x = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$



$$\left\| \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \right\| = \sqrt{6}$$

$$= \left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|$$

We just saw: least squares soln is $(5,-3)$

$$\longrightarrow \quad y = -3x + 5$$

Best fit lines

Poll

What does the best fit line minimize?

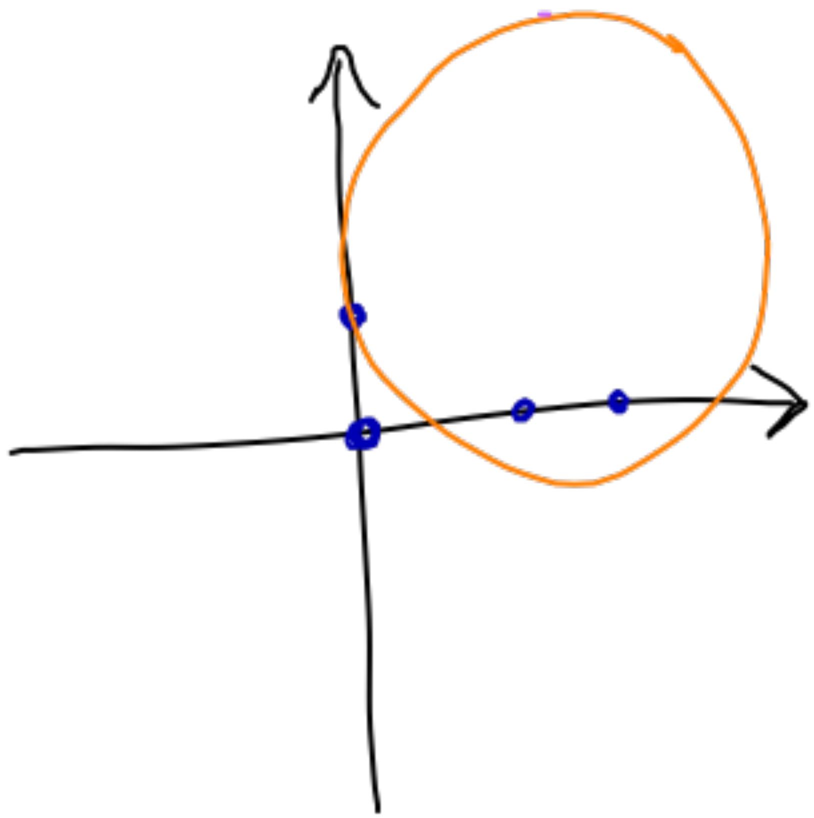
1. the sum of the squares of the distances from the data points to the line
2. the sum of the squares of the vertical distances from the data points to the line
3. the sum of the squares of the horizontal distances from the data points to the line
4. the maximal distance from the data points to the line

Least Squares Problems

More applications

Determine the least squares problem $Ax = b$ to find the best fit circle/ellipse for the points:

$$(0,0), (2,0), (3,0), (0,1)$$



General Eqn for circle:

$$C + x^2 + y^2 + Dx + Ey = 0$$

$$\begin{aligned} C &= 0 \\ C + 4 + 2D &= 0 \\ C + 9 + 3D &= 0 \\ C + 1 + E &= 0 \end{aligned} \quad \rightsquigarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ -4 \\ -9 \\ -1 \end{pmatrix}$$

Gauss invented the method of least squares to predict the orbit of the asteroid Ceres as it passed behind the sun in 1801.

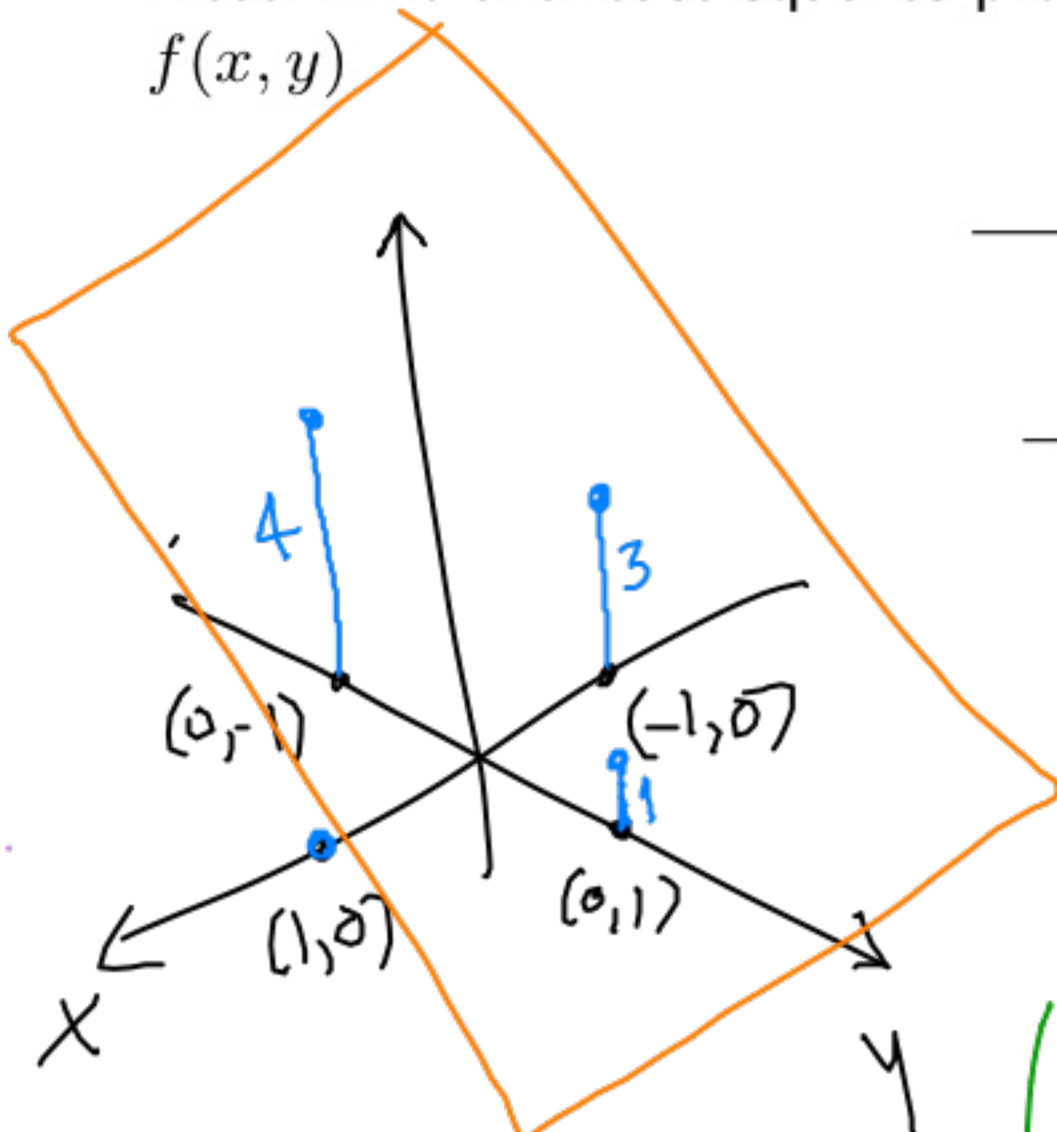
Determine the least squares problem $Ax = b$ to find the best parabola (quadratic function of x) for the points:

$$(0,0),(2,0),(3,0),(0,1)$$

Least Squares Problems

More applications

Determine the least squares problem $Ax = b$ to find the best fit linear function $f(x, y)$



x	y	$f(x,y)$
1	0	0
0	1	1
-1	0	3
0	-1	4

General eqn for a linear fn of x,y:

$f(x,y) = Ax + By + C$

$A + C = 0$

$B + C = 1$

$-A + C = 3$

$-B + C = 4$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}$$