Announcements April 4

- WebWork 5.3 and 5.5 due Thursday Supplemental 5.5
- Homework 7 due Friday April 8
- Midterm 3 in class Friday April 8 on Chapter 5
- Office Hours Wed 2-3 (and more on Thu).
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
    - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
    - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
    - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 5.5

Complex Eigenvalues

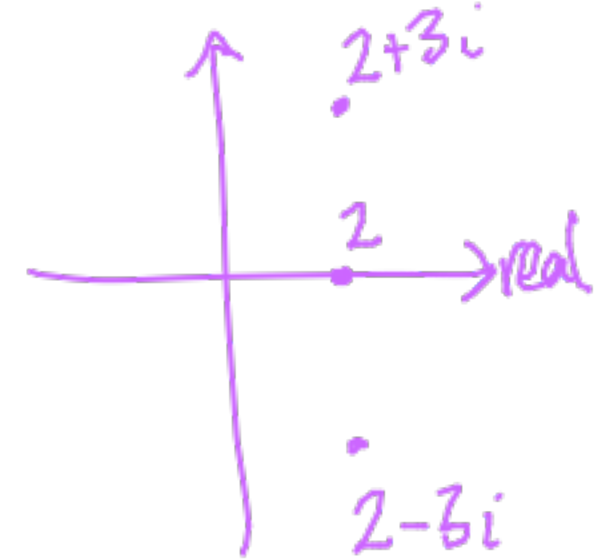
A matrix with an eigenvector

Find the eigenvectors and eigenvalues of:

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

$$\overline{a+bi} = a-bi$$

$$\overline{-2+0i} = -2$$



Eigenvalues

$$(1-\lambda)(3-\lambda)+2=0$$

$$\lambda^2-4\lambda+5=0$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

Eigenvectors $\boxed{2+i}$

$$\begin{pmatrix} 1-(2+i) & -2 \\ 1 & 3-(2+i) \end{pmatrix} = \begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} -1-i & -2 \\ 0 & 0 \end{pmatrix}. \quad \text{eigenvector } \begin{pmatrix} -2 \\ 1+i \end{pmatrix}$$

$\boxed{2-i}$ $\begin{pmatrix} -2 \\ 1-i \end{pmatrix}$

What do complex eigenvalues mean?

Fact. If an $n \times n$ matrix A has a complex eigenvalue there is a 2D plane in $\mathbb{R}^n$ where A is (similar to) the product of a rotation and a dilation.

Here is the actual statement for 2×2 matrices:

Theorem. Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ (where $b \neq 0$) and associated eigenvector v. Then

$$A = PCP^{-1}$$

A similar to C.

looks like.

where

$$P = (\operatorname{Re} v \quad \operatorname{Im} v) \quad \text{and} \quad C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Diagonalization

$A = CDC^{-1}$

$A = C\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}C^{-1}$

C is the composition of a rotation by θ and scaling by r.

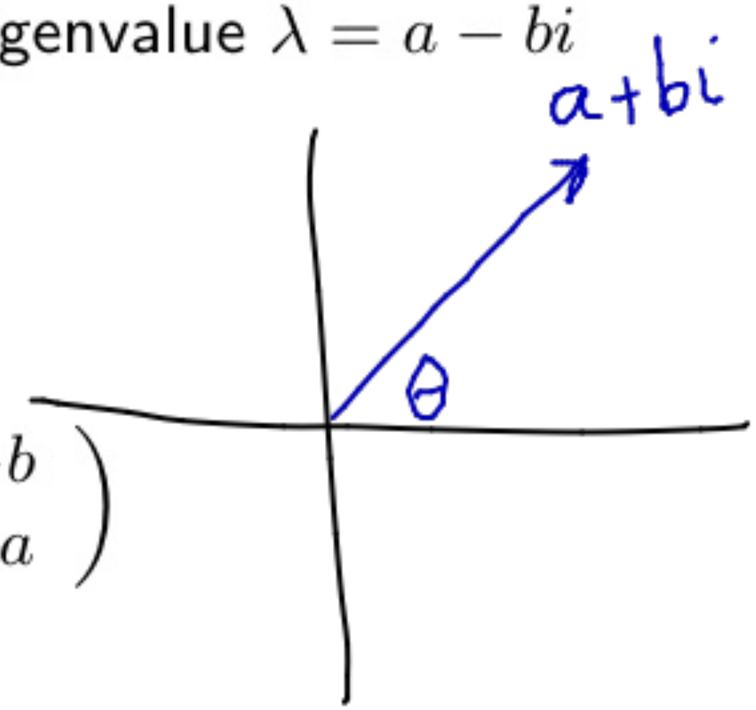
What do complex eigenvalues mean?

Theorem. Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ (where $b \neq 0$) and associated eigenvector v. Then

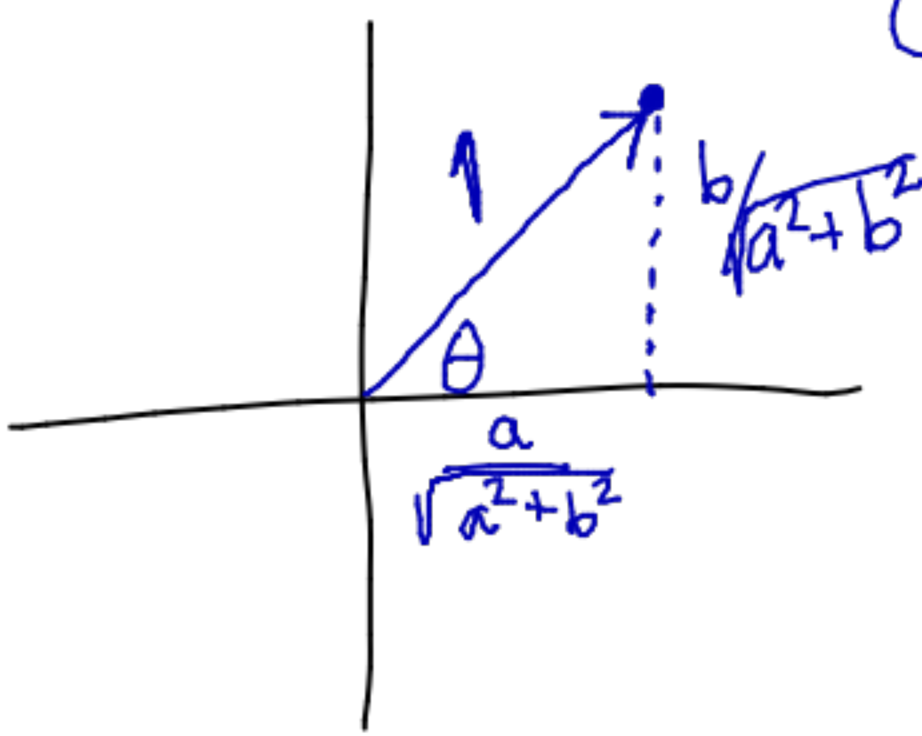
$$A = PCP^{-1}$$

where

$$P = (\mathrm{Re}\, v \quad \mathrm{Im}\, v) \quad \text{and} \quad C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$



C is the composition of a rotation by θ and scaling by r. Why?



$$C = \sqrt{a^2+b^2} \begin{pmatrix} \frac{a}{\sqrt{a^2+b^2}} & -\frac{b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}$$

$$= \sqrt{a^2+b^2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Scale rotation.

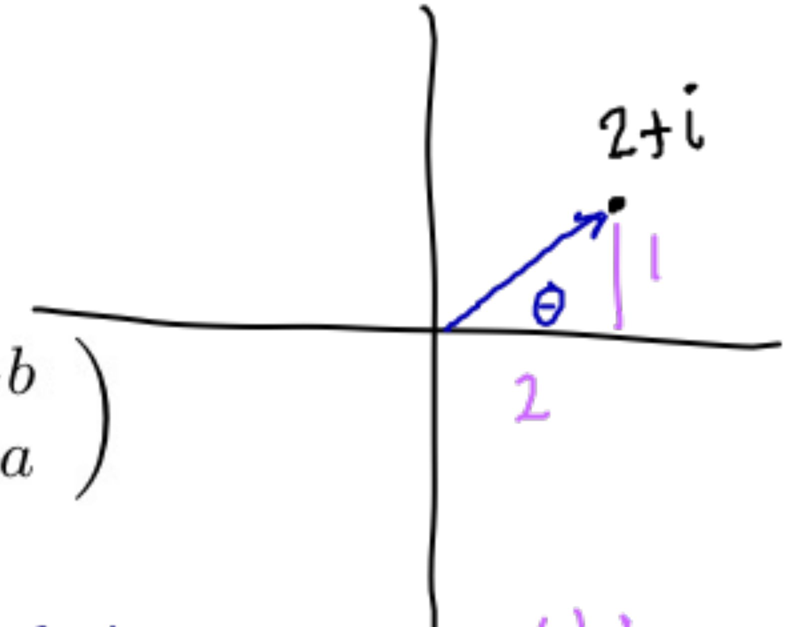
What do complex eigenvalues mean?

Theorem. Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ (where $b \neq 0$) and associated eigenvector v. Then

$$A = PCP^{-1}$$

where

$$P = (\operatorname{Re} v \quad \operatorname{Im} v) \quad \text{and} \quad C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$



Example. Find C and P when

$$A = \cancel{\begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}} \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

$\tan^{-1}\left(\frac{b}{a}\right)$

<u>eigenvalue</u> $2 - i \rightsquigarrow C = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

$a = 2, b = 1$

rotation by: $\tan^{-1}(1/2)$

scale by: $\sqrt{2^2 + 1^2} = \sqrt{5}$

<u>eigenvector</u> $\begin{pmatrix} -2 \\ 1-i \end{pmatrix} \rightsquigarrow P = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}$

$$A = PCP^{-1}$$

Three pictures

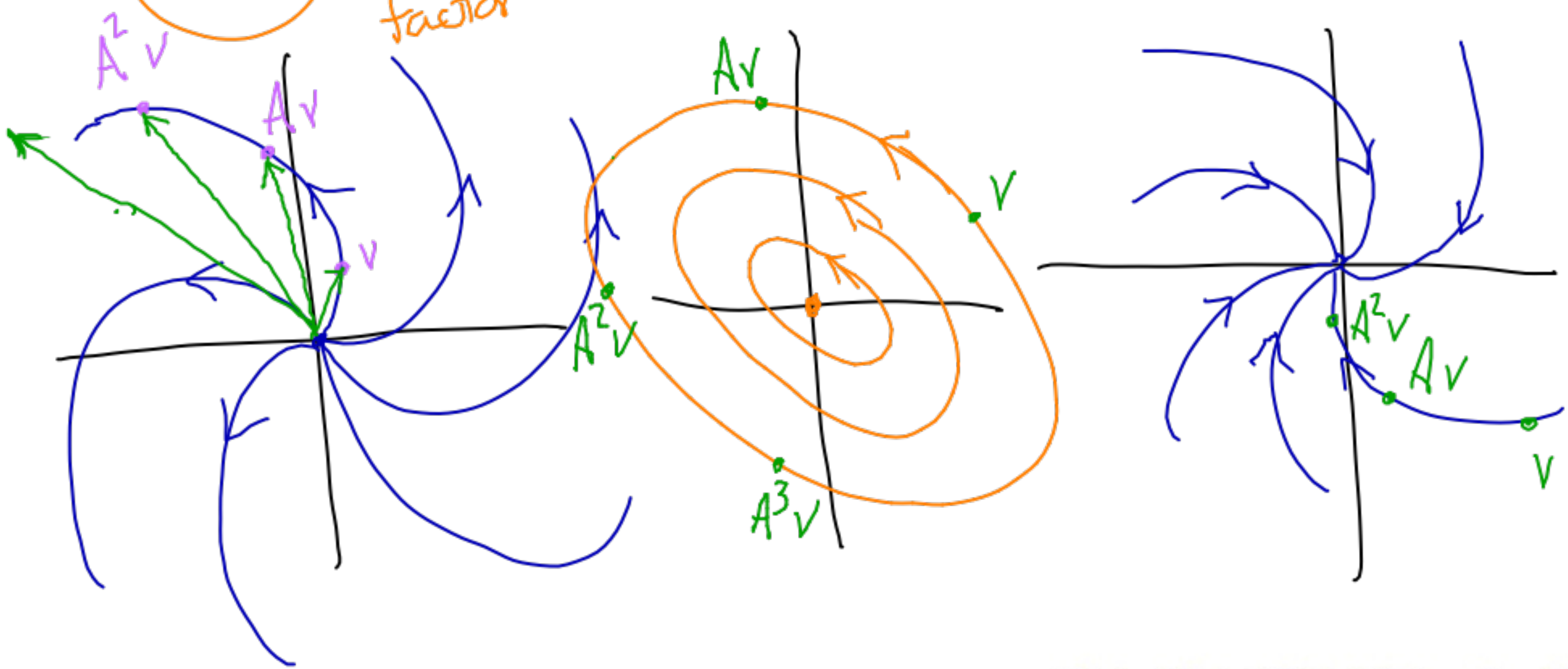
There are three possible pictures for the action on $\mathbb{R}^2$ of a 2×2 matrix with complex eigenvalues.

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix} \qquad A = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\lambda = 1 + i \qquad \lambda = 3/5 + 4/5i \qquad \lambda = 1/2 + 1/2i$$

$$|\lambda| > 1 \qquad |\lambda| = 1 \qquad |\lambda| < 1$$

scale factor



What about the higher dimensional case?

Theorem. Let A be a real $n \times n$ matrix. Suppose that for each (real or complex) eigenvalue the dimension of the eigenspace equals the algebraic multiplicity. We can find an block diagonalization

$$A = PCP^{-1}$$

where P and C are found as follows:

1. C is *block diagonal* where the blocks are 1×1 blocks containing the real eigenvalues (with multiplicity) or 2×2 blocks containing the matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for each complex eigenvalue $a - bi$ (with multiplicity).
2. The columns of P form bases for the eigenspaces for the real eigenvectors or come in pairs $(\mathrm{Re}\, v \quad \mathrm{Im}\, v)$ for the complex eigenvectors.

real eigenval

$P = \Big(\quad \Big)$ $\qquad C = \begin{pmatrix} \boxed{2} & \\ & \boxed{\begin{matrix} a & -b \\ b & a \end{matrix}} \end{pmatrix}$

2-eigenvector

Re & Im part of eigenvector for complex eigenvalue.

A 3×3 example

Find the block diagonalization of:

$$A = \begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

A 3×3 example

What does A do to $\mathbb{R}^3$? Draw a picture!

$$A = \begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$