Announcements Feb 17

- WebWork 2.1 and 2.2 due Thursday
- Homework 4 due in class Friday
- Midterm 2 in class Friday Mar 11 on Chapters 2 & 3
- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Baishen Wed 4-5, Matt Thu 3-4, Shivang Fri 10:30-11 + 12:30-1
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6
Section 2.2

The Inverse of a Matrix

\[ 5x = 35 \]

\[ Ax = b \]
Inverses

\[ A = n \times n \text{ matrix.} \]

\( A \) is invertible (or nonsingular) if there is a matrix \( B \) with

\[
\begin{align*}
AB &= BA \\
&= I_n
\end{align*}
\]

\( B \) is called the inverse of \( A \) and is written \( A^{-1} \)

Example:

\[
\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}
\]

\[
\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
The $2 \times 2$ Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - bc$ is the determinant of $A$.

\[ \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2 \]

Fact. If $\det(A) \neq 0$ then $A$ is invertible and $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $\det(A) = 0$ then $A$ is not invertible.

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$
Solving Linear Systems via Inverses

**Fact.** If $A$ is invertible, then $Ax = b$ has exactly one solution, namely

$$A^{-1}Ax = A^{-1}b$$

$$I_n x = A^{-1}b$$

$$x = A^{-1}b$$

$$I_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**Example.** Solve

$$2x + 3y + 2z = 1$$

$$x + 3z = 1$$

$$2x + 2y + 3z = 1$$

Using

$$A^{-1} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

$$Ax = b$$

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x = A^{-1}b = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

If we change

$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

to

$$b = \begin{pmatrix} 7 \\ -13 \\ -19 \end{pmatrix}$$

just do $A^{-1}b = \begin{pmatrix} 7 \\ -13 \\ -19 \end{pmatrix}$
Some Facts

Say that $A$ and $B$ are invertible $n \times n$ matrices.

- $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$
- $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
- $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$

$$\begin{align*}
(AB)(B^{-1}A^{-1}) &= A \cdot I_n \cdot A^{-1} \quad \text{= } AA^{-1} = I_n \\
(A^{-1}A)^T &= I_n^T = I_n \\
(ABC)(C^{-1}B^{-1}A^{-1}) &= I \\
(XY)^T &= Y^T \cdot X^T
\end{align*}$$
An Algorithm for Finding $A^{-1}$

Suppose $A = n \times n$ matrix.

- Row reduce $(A \mid I_n)$
- If reduction has form $(I_n \mid B)$ then $A$ is invertible and $B = A^{-1}$.
- Otherwise, $A$ is not invertible.

**Example.** Find

$$
\begin{pmatrix}
1 & 0 & 4 \\
0 & 1 & 2 \\
0 & -3 & -4 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
$$

Thus, the inverse is

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & -6 & -2 \\
0 & -2 & -1 \\
0 & 3/2 & 1/2 \\
\end{pmatrix}
$$
Why Does This Work?

First answer: we can think of the algorithm as simultaneously solving

\[ Ax_1 = e_1 \]
\[ Ax_2 = e_2 \]

and so on. But the columns of \( A^{-1} \) are \( A^{-1}e_i \), which is \( x_i \).

There is another explanation, which uses elementary matrices.
Elementary matrices

An elementary matrix, \( E \), is one that differs by \( I_n \) by one row operation.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

**Fact.** If \( E \) is an elementary matrix for some row operation, then \( EA \) differs from \( A \) by same row operation.

Why? Check for each type.

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
= \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
= \begin{pmatrix} 7 & 10 \\ 3 & 4 \end{pmatrix}
\]

**Fact.** Elementary matrices are invertible.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{pmatrix}^{-1}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1/2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}^{-1}
= \begin{pmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}^{-1}
= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}^{-1}
= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}^{-1}
= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
Elementary matrices

Observation. An $n \times n$ matrix $A$ is invertible iff it is row equivalent to $I_n$. In this case, the sequence of row operations taking $A$ to $I_n$ also takes $I_n$ to $A^{-1}$. This gives us a second explanation of the algorithm.

Why is it true?

Row ops taking $A$ to $I_n$

$$(E_k \cdots E_2 E_1)A = I_n$$

$$(E_k \cdots E_2 E_1)AA^{-1} = I_n A^{-1}$$

$$(E_k \cdots E_1)I_n = A^{-1}$$

$$A \mid I_n \sim (I_n \mid A^{-1})$$
Structural Engineering

Suppose we put 3 downward forces on an elastic beam.

By Hooke’s law, the vertical displacements at those three points $y_1, y_2, y_3$ are given by a linear transformation.

$$
\begin{align*}
\begin{bmatrix}
  f_1 \\
  f_2 \\
  f_3 \\
\end{bmatrix} &= 
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
\end{bmatrix}
\end{align*}
$$

If we want to achieve a certain displacement, use $D^{-1}$ to find the required forces.

$$
D^{-1} \begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
\end{bmatrix} = \begin{bmatrix}
  f_1 \\
  f_2 \\
  f_3 \\
\end{bmatrix}
$$