Announcements Feb 17

- WebWork 2.1 and 2.2 due Thursday
- Homework 4 due in class Friday
- Midterm 2 in class Friday Mar 11 on Chapters 2 & 3
- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Baishen Wed 4-5, Matt Thu 3-4, Shivang Fri 10:30-11 + 12:30-1
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 2.2

The Inverse of a Matrix

$5x = 35$

$Ax = b$

Inverses

$A = n \times n$ matrix.

A is invertible (or nonsingular) if there is a matrix B with

$$AB = BA = I_n$$

B is called the inverse of A and is written A^{-1}

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The 2×2 Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - bc$ is the determinant of A.

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$$

Fact. If $\det(A) \neq 0$ then A is invertible and $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $\det(A) = 0$ then A is not invertible.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

Solving Linear Systems via Inverses

Fact. If A is invertible, then $Ax = b$ has exactly one solution, namely

$$A^{-1}Ax = A^{-1}b$$
$$I_n x = A^{-1}b$$
$$\boxed{x = A^{-1}b}$$

$$I_n = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

Example. Solve

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 1 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

$$Ax = b \qquad A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

$$x = A^{-1}b = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ +1 \\ 1 \end{pmatrix}$$

If we change $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ *to* $\begin{pmatrix} 7 \\ 13 \\ -19 \end{pmatrix}$ *just do* $A^{-1}\begin{pmatrix} 7 \\ 13 \\ -19 \end{pmatrix}$

Some Facts

Say that A and B are invertible $n \times n$ matrices.

- A^{-1} is invertible and $(A^{-1})^{-1} =$ A
- AB is invertible and $(AB)^{-1} =$ ~~$A^{-1}B^{-1}$~~ $B^{-1}A^{-1}$
- A^T is invertible and $(A^T)^{-1} =$ $(A^{-1})^T$

$(AB)(B^{-1}A^{-1})$
$= A I_n A^{-1}$
$= AA^{-1} = I_n$

$A^T (A^{-1})^T$
$= (A^{-1}A)^T$
$= I_n^T = I_n$

Q. What is $(ABC)^{-1}$?

$C^{-1}B^{-1}A^{-1}$

$(ABC)(C^{-1}B^{-1}A^{-1}) = I$

$(XY)^T = Y^T X^T$

An Algorithm for Finding A^{-1}

Suppose $A = n \times n$ matrix.

- Row reduce $(A \mid I_n)$
- If reduction has form $(I_n \mid B)$ then A is invertible and $B = A^{-1}$.
- Otherwise, A is not invertible.

Example. Find $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}^{-1}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right)$$

Thus, the inverse is

Why Does This Work?

First answer: we can think of the algorithm as simultanenously solving

$$Ax_1 = e_1$$
$$Ax_2 = e_2$$

and so on. But the columns of A^{-1} are $A^{-1}e_i$, which is x_i.

There is another explanation, which uses elementary matrices.

Elementary matrices

An elementary matrix, E, is one that differs by I_n by one row operation.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Fact. If E is an elementary matrix for some row operation, then EA differs from A by same row operation.

Why? Check for each type.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 3 & 4 \end{pmatrix}$$

Fact. Elementary matrices are invertible.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Elementary matrices

Observation. An $n \times n$ matrix A is invertible iff it is row equivalent to I_n. In this case, the sequence of row operations taking A to I_n also takes I_n to A^{-1}. This gives us a second explanation of the algorithm.

$(A \mid I_n) \rightsquigarrow (I_n \mid A^{-1})$

Why is it true?

Row ops taking A to I_n

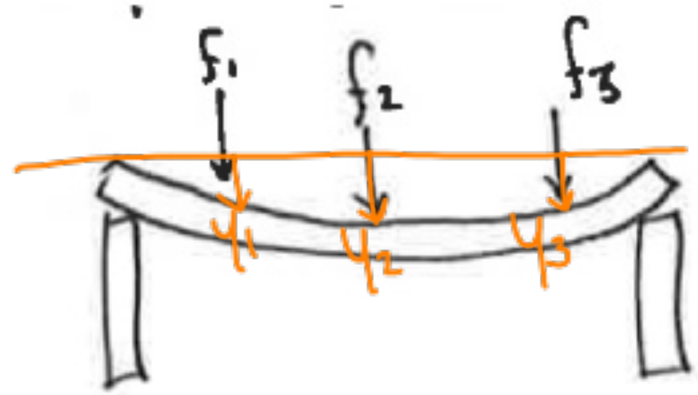
$$(E_k \cdots E_2 E_1) A = I_n$$

$$(E_k \cdots E_2 E_1) A A^{-1} = I_n A^{-1}$$

$$(E_k \cdots E_1) I_n = A^{-1}$$

Structural Engineering

Suppose we put 3 downward forces on an elastic beam.



By Hooke's law, the vertical displacements at those three points y_1, y_2, y_3 are given by a linear transformation.

stiffness matrix →

$$D \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

If we want to achieve a certain displacement, use D^{-1} to find the required forces.

$$D^{-1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$