Announcements Feb 22

- WebWork 2.3 and 2.5 due Thursday
- Homework 5 due in class Friday
- Quiz 5 on 2.3 and 2.5 in class Friday
- Midterm 2 in class Friday Mar 11 on Chapters 2 & 3
- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Baishen Wed 4-5, Matt Thu 3-4, Shivang Fri 10:30-11 + 12:30-1
- Math Lab, Clough 280
    - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
    - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
    - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 2.5

~~Matrix~~ Decompositions

LU

LU Decomposition

Summary

Recall: If we want to solve $Ax = b$, we can:

- row reduce $(A|b)$, or
- find A^{-1}.

Today: the method of LU decomposition.

Computational complexity of row reduction: $n^4/3$
Computational complexity of LU decomposition: $4n^3/3$

LU Decomposition

Outline

- LU decompositions
- Using LU decompositions to solve $Ax = b$
- Finding LU decompositions: an example when A is square
- Finding LU decompositions: an example when A is not a square
- Application to electrical engineering (circuits)
- What do do when there are row swaps

LU Decomposition

An LU factorization of $A = m \times n$ is an expression

$$A = LU$$

where

- $L = m \times m$ unit lower triangular matrix
- $U = m \times n$ echelon form of A

U = "upper"

unit: all diags are 1.

Lower triang: 0's above diag

not 0

0's

e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 5 & 1 \end{pmatrix}$

Example.

$$\begin{pmatrix} 3 & 1 \\ -6 & -4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}}_{U}$$

LU Decomposition

Solving $Ax = b$

To solve $Ax = b$, we write

$$A = m \times n$$

$$\begin{aligned} Ax &= b \\ LUx &= b \end{aligned}$$

and

1. solve $Ly = b$, to obtain y,
2. solve $Ux = y$ to obtain x.

$$\begin{tikzcd} \boxed{x} \\[-2em] \mathbb{R}^n \arrow[rr, "A = LU"] \arrow[dr, swap, "U"] & & \mathbb{R}^m \\[-2em] & \mathbb{R}^m \arrow[ur, swap, "L"] & \end{tikzcd}$$

$$Ux = \boxed{y}$$

$$\boxed{b} = Ax = L(Ux)$$

Both steps simple since U, L are.

This approach uses only back substitution, *not* elimination.

LU Decomposition

Solving $Ax = b$

After writing $A = LU$, two steps:

1. solve $Ly = b$, to obtain y,
2. solve $Ux = y$ to obtain x.

Example.

$$\begin{pmatrix} 3 & 1 \\ -6 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

Solve $Ax = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$

Step 1. Solve $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$

$\rightsquigarrow y_1 = 5$, $-2y_1 + y_2 = -8$

$\rightsquigarrow -10 + y_2 = -8$

$y_2 = 2$

$\rightsquigarrow y = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

Step 2. $\begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$\rightsquigarrow x_2 = -1$

$\rightsquigarrow 3x_1 + x_2 = 5$

$3x_1 - 1 = 5 \rightsquigarrow x_1 = 2$

$\boxed{x = \begin{pmatrix} 2 \\ -1 \end{pmatrix}}$

Finding the LU Decomposition

We do row operations on A, using only row replacements, and doing them in the standard order. Then U is the reduced matrix and L records the negatives of the row operations.

$$A = \begin{pmatrix} 6 & 0 & 2 \\ 24 & 1 & 8 \\ -12 & 1 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 6 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 6 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

order 1 2 3

U

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{+4} & 1 & 0 \\ \boxed{-2} & \boxed{+1} & 1 \end{pmatrix}$$

Finding the LU Decomposition

Why Does This Method Work?

Row operations are elementary matrices, so $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{aligned} \underline{E_4 E_3 E_2 E_1} A &= U \\ A &= \underline{(E_4 E_3 E_2 E_1)}^{-1} U \\ &= \left(E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}\right) U \\ &= LU \end{aligned}$$

Now use these facts

- each E_i is unit lower triangular, so each E_i^{-1} is as well
- the product of unit lower triangular matrices is lower triangular

Using LU to solve a linear system

Step 1. $Ly = b$
Step 2. $Ux = y$.

We found:

$$A = \begin{pmatrix} 6 & 0 & 2 \\ 24 & 1 & 8 \\ -12 & 1 & -3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 6 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{U}$$

Use this to solve $Ax = \begin{pmatrix} 4 \\ 19 \\ -6 \end{pmatrix}$.

Step 1. $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 19 \\ -6 \end{pmatrix}$

$y_1 = 4$

$4y_1 + y_2 = 19 \rightsquigarrow y_2 = 3$

$-2y_1 + y_2 + y_3 = -6 \rightarrow y_3 = -1$

$y = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$

Step 2. $\begin{pmatrix} 6 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$

$x_3 = -1$

$x_2 = 3$

$6x_1 + 2x_3 = 4 \rightarrow x_1 = 1$

$$\boxed{x = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}}$$

Finding the LU Decomposition

A non-square example

$$A = \begin{pmatrix} -2 & 1 & 3 \\ -4 & 4 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 & 1 & 3 \\ 0 & 2 & -5 \end{pmatrix}$$

U

$L \quad U$

$m \times m \quad 2 \times 3$

$$L = \begin{pmatrix} 1 & 0 \\ \boxed{+2} & 1 \end{pmatrix}$$

Solve $Ax = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ Step 1.

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$y_1 = 2$

$2y_1 + y_2 = 1 \rightsquigarrow y_2 = -3$

Step 2

$$\begin{pmatrix} -2 & 1 & 3 \\ 0 & 2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$x_3 = \text{free}$

$x_2 = \frac{-3+5x_3}{2}$

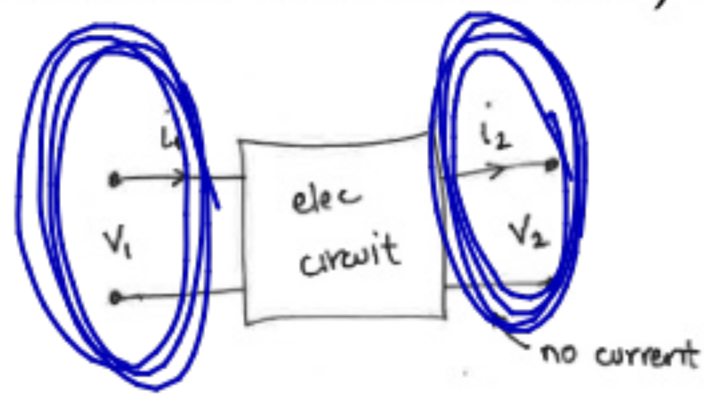
$-2x_1 + x_2 + 3x_3 = 2$

$x_1 = \left[2 - 3x_3 - \left(\frac{-3+5x_3}{2}\right)\right] / -2$

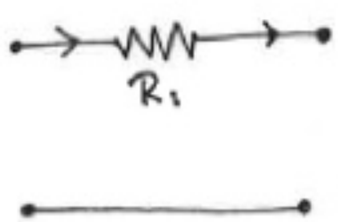
$$\begin{pmatrix} \cancel{1}\ -7/4 \\ -3/2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} \boxed{11/4} \\ 5/2 \\ 1 \end{pmatrix}$$

Application to Electrical Engineering

In an electrical circuit, current i and voltage v often change by a linear transformation (by Ohm's law and Kirchoff's law).

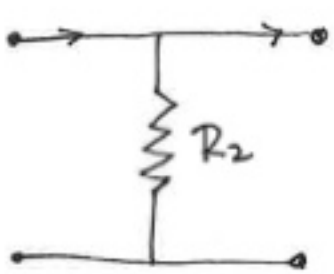


So $A\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$ for some transfer matrix A.



series circuit

$$A = \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix}$$

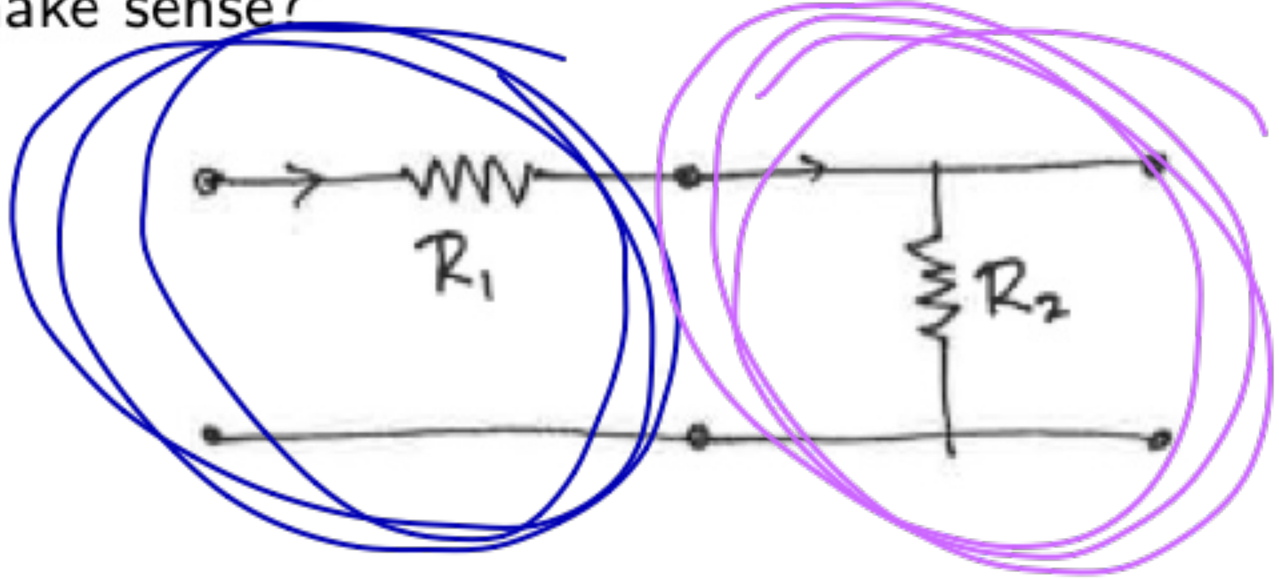


shunt circuit

$$A = \begin{pmatrix} 1 & 0 \\ -1/R_2 & 1 \end{pmatrix}$$

Application to Electrical Engineering

If we string these small circuits together we get a ladder circuit. The transfer matrix for the ladder circuit is the product of the matrices for the components. Why does this make sense?



The transfer matrix is:

$$\begin{pmatrix} 1 & 0 \\ -1/R_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -R_1 \\ -1/R_2 & 1 + R_1/R_2 \end{pmatrix}$$

L U

Can you make a ladder circuit whose transfer matrix is

$$\begin{pmatrix} 1 & -8 \\ -0.5 & 5 \end{pmatrix} ?$$

If row swaps are needed, we introduce a permutation matrix, P, so that

$$PA = LU$$

Example. $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 0 \end{pmatrix} \rightsquigarrow$ proceed as before.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$