Announcements Feb 29

I have papers thru Week 4

- WebWork 2.8 and 2.9 due Thursday
- Homework 6 due Friday
- Quiz 6 on 2.8 and 2.9 in class Friday
- Midterm 2 in class Friday Mar 11 on Chapters 2 & 3
- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 2.9

Dimension and Rank

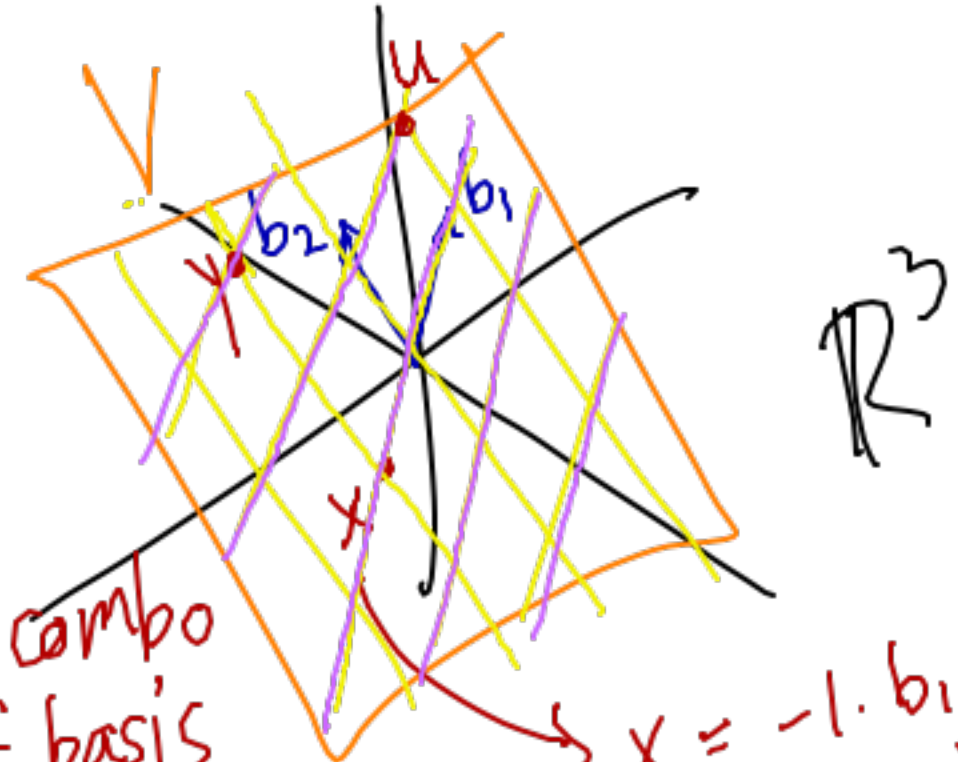
Bases as Coordinate Systems

V = subspace of $\mathbb{R}^n$

$B = \{b_1, b_2, \dots, b_k\}$ is a basis for V

lin ind vectors that span V

x a vector in V



Then we can write x uniquely as

$c_1 b_1 + \cdots + c_k b_k$

lin combo of basis vectors

$x = -1 \cdot b_1 + 0 \cdot b_2$

We write

$$[x]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

$[x]_B = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$[y]_B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$[u]_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

These are the B-coordinates of x.

Bases as Coordinate Systems

Example

Say $b_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $b_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$B = \{b_1, b_2\}$

$V = \text{Span}\{b_1, b_2\}$.

Q. Verify that B is a basis for V and find the B-coordinates of $x = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$

B spans V by defn of V

B lin ind since b_1, b_2 not multiples of eachother.

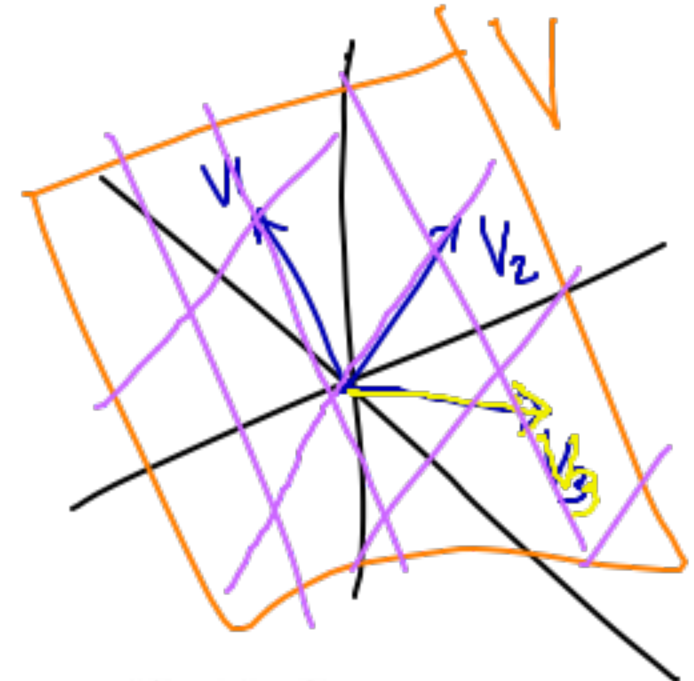
$c_1 b_1 + c_2 b_2 = x$ or $c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

$[x]_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Bases as Coordinate Systems

Example



Say $v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}$

$$V = \mathrm{Span}\{v_1, v_2, v_3\}.$$

Q. Find a basis for V and find the B-coordinates of $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$

Finding the basis:

V is column space of

$$\begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & 8 \\ 2 & 1 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 & 2 \\ 0 & 5/2 & 5 \\ 0 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

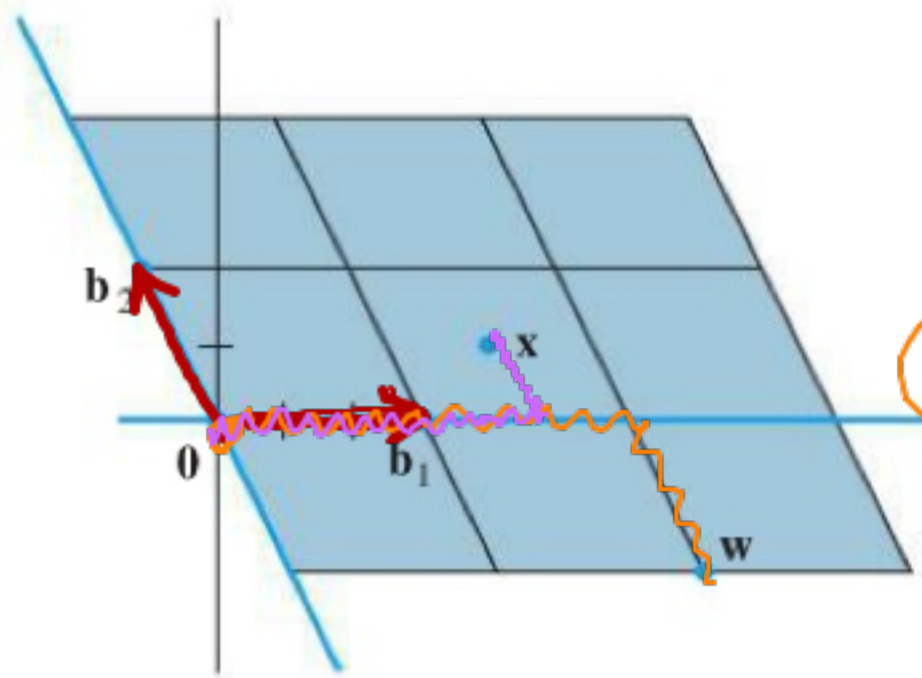
$$\rightsquigarrow \begin{pmatrix} \boxed{2} & -1 & 2 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{basis} = \{v_1, v_2\}.$$

To find $[x]_B$: $\left(\begin{array}{cc|c} 2 & -1 & 4 \\ 3 & 1 & 11 \\ 2 & 1 & 8 \end{array}\right) \xrightarrow[\text{reduce}]{\text{row}} [x]_B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Bases as Coordinate Systems

Consider the following basis for $\mathbb{R}^2$:

$$B = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$



$[w]_B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$[x]_B = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$

$[2x]_B = 2[x]_B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\frac{3}{2} b_1 + \frac{1}{2} b_2 = x$

$3 b_1 + 1 \cdot b_2 = 2x$

Use the figure to estimate the B-coordinates of

$$w = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \text{ and } x = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

usual coords
(e_1, e_2)

Rank Theorem

Define:

$$\text{rank}(A) = \dim \text{Col}(A) = \text{\# cols w/ pivots}$$
$$\dim \text{Nul}(A) = \text{\# cols w/o pivots}$$

Rank Theorem

If A is an $m \times n$ matrix, then $\text{rank}(A) + \dim \text{Nul}(A) = n$ = # cols of A

So: If have $Ax=b$

have degrees of freedom in choosing b

& deg's of freedom in choosing x.

Example.

If $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, then $\text{rank}(A) = 1$ and $\dim \text{Nul}(A) = 2$

But they add to n.

Poll

Poll

If A and B are 3×3 matrices, and $\mathrm{rank}(A) = \mathrm{rank}(B) =$ 2 then what are the possible values of $\mathrm{rank}(AB)$?

a) 0

b) 1

c) 2

~~d)~~ 3

~~e)~~ 4

find examples

ABv

v

w

$$\underbrace{\underbrace{\mathbb{R}^3 \xrightarrow{T_B} \mathbb{R}^3}_{\text{not 1-1}} \xrightarrow{T_A} \mathbb{R}^3}_{\text{not 1-1}}$$

Two More Theorems

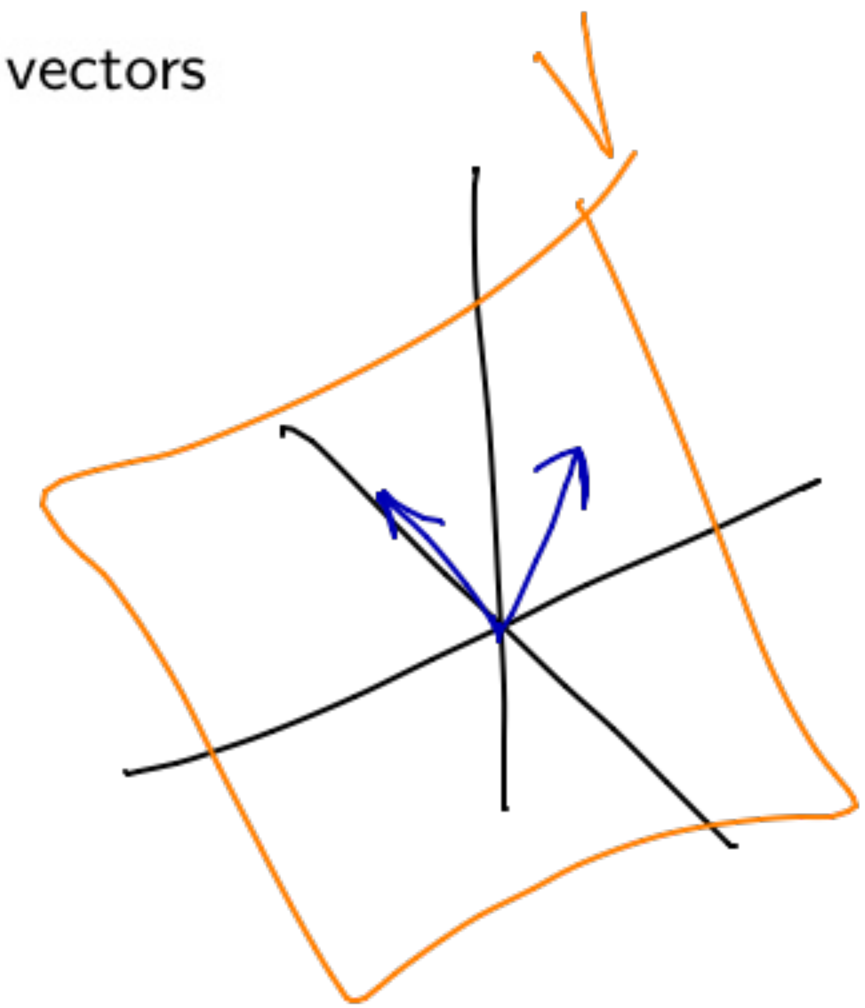
Basis Theorem
If V is a k-dimensional subspace of $\mathbb{R}^n$, then

- any k linearly independent vectors of V form a basis for V
- any k vectors that span V form a basis for V

In other words if a set has two of these three properties, it is a basis:

spans V, linearly independent, k vectors

defn of
basis.



Two More Theorems

Invertible Matrix Theorem

(a) A is invertible

$\vdots$

(m) cols of A form a basis for $\mathbb{R}^n$

(n) $\mathrm{Col}(A) = \mathbb{R}^n$

(o) $\dim \mathrm{Col}(A) = n$

(p) $\mathrm{rank}(A) = n$

(q) $\mathrm{Nul}(A) = \{0\}$

(r) $\dim \mathrm{Nul}(A) = 0$