Announcements Feb 3

- WebWork 1.5 due Thursday
- Written Homework 3 due Friday
- Quiz 3 on Friday on Section 1.5
- Midterm 1 in class next week Friday Feb 12
- My Office Hours Tuesday and Wednesday 2-3, after class, and by appointment
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Baishen Wed 4-5, Matt Thu 3-4, Shivang Fri 10:30-11 + 12:30-1
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 1.8

Introduction to Linear Transformations

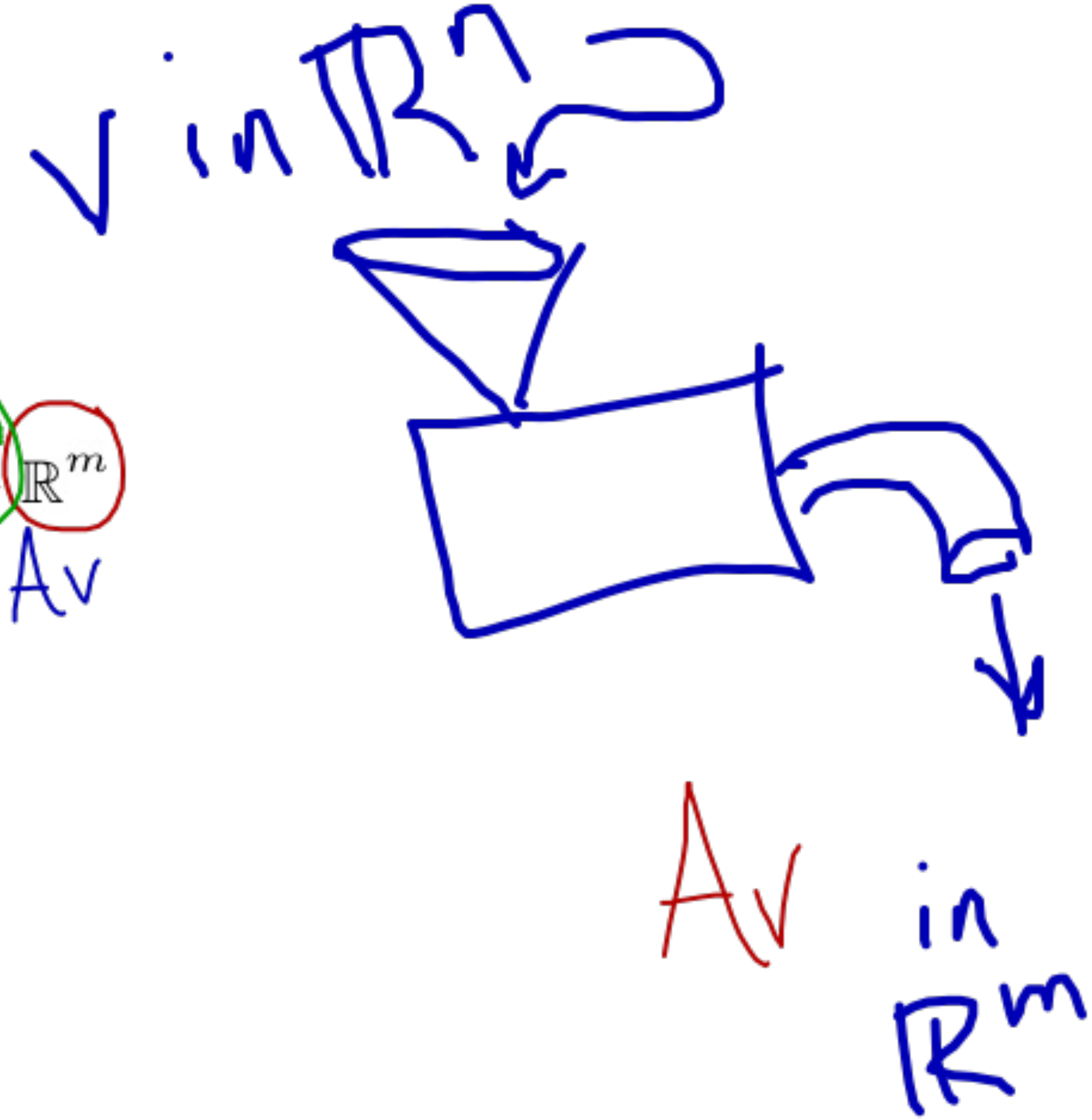
From matrices to functions

Let A be an $m \times n$ matrix.

We define a function

$$\underset{m\times n}{\begin{pmatrix} \quad A \quad \end{pmatrix}}\underset{n\times 1}{\begin{pmatrix} v \end{pmatrix}} = \underset{m\times 1}{\begin{pmatrix} Av \end{pmatrix}} \qquad \begin{aligned} T_A &: \mathbb{R}^n \to \mathbb{R}^m \\ T_A(v) &= Av \end{aligned}$$

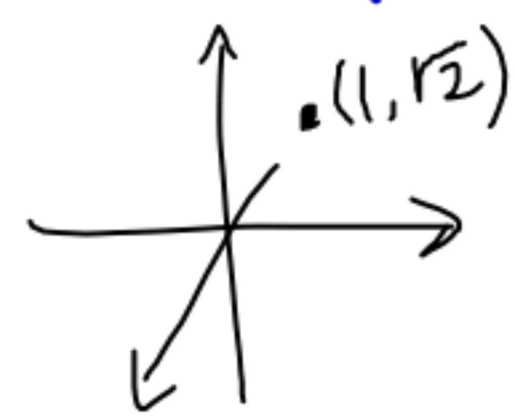
This is called a matrix transformation.



The domain of T_A is $\mathbb{R}^n$

The co-domain/target of T_A is $\mathbb{R}^m$

The range/image of T_A is all outputs:
span of cols of A



This gives us another point of view of $Ax = b$.

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$.

3×2

Domain $\mathbb{R}^2$

Target $\mathbb{R}^3$

What is $T_A(u)$?

$$Au = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$$

Find v so that $T_A(v) = b$

↳ in $\mathbb{R}^2$

Want v so:

$$Av = b$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} v = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix} \qquad \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right)$$

$y = 5$

$x = 2$

$v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

Find c so there is no v with $T_A(v) = c$

or Find c not in range of T_A

or Find c not in span of cols of A

$$a\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ b \\ a+b \end{pmatrix}$$

$c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ since $1 \neq 3$

Dynamical systems

When A is a square matrix ($m = n$) we can think of

$$T_A : \mathbb{R}^n \to \mathbb{R}^n$$

as doing something to $\mathbb{R}^n$.

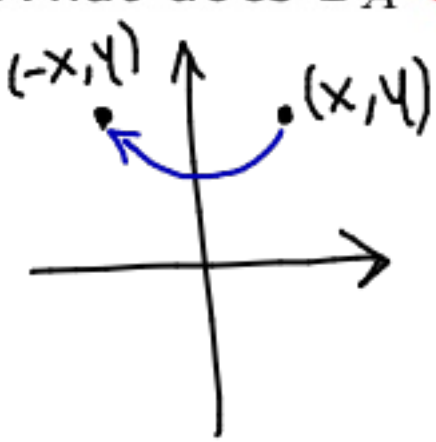
Example. If

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$T_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

What does T_A do to $\mathbb{R}^2$?

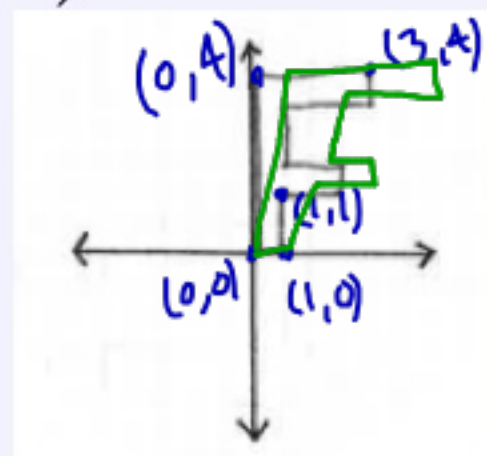


reflection about y-axis.

Clicker

Poll

What does $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ do to this letter F?



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$$

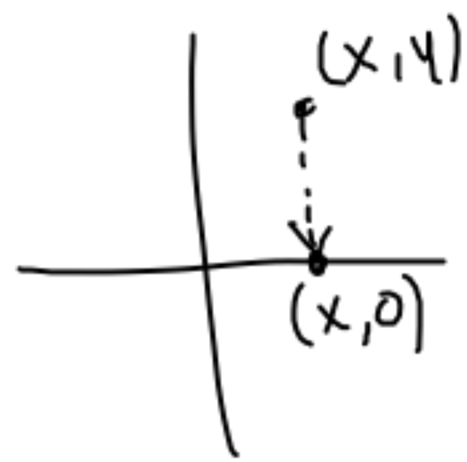
Examples in $\mathbb{R}^2$

What do these matrices do to $\mathbb{R}^2$?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

reflects about $y=x$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$



projection to x-axis

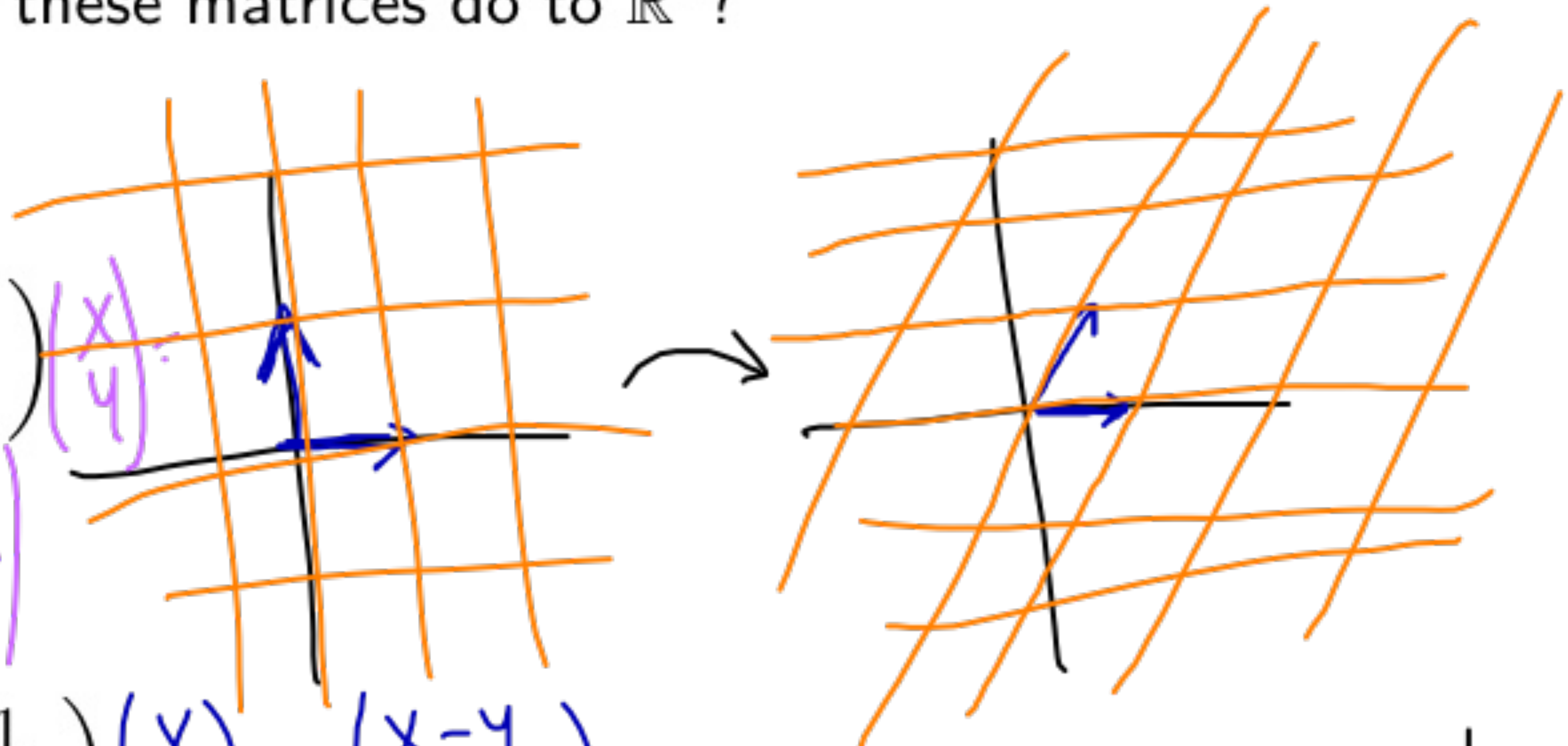
$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$$

dilation by 3
or scale by 3

Examples in $\mathbb{R}^2$

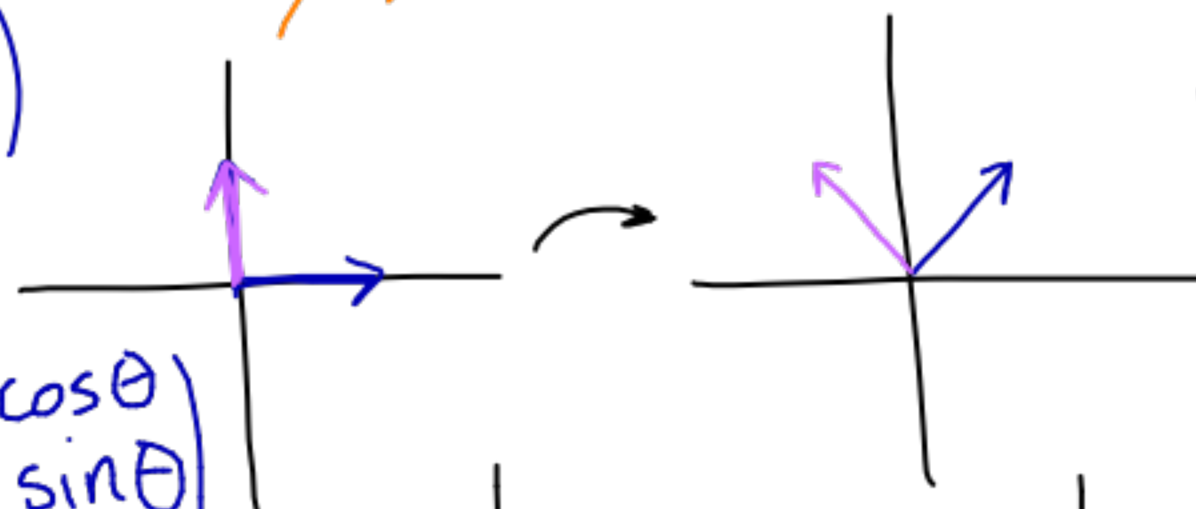
What do these matrices do to $\mathbb{R}^2$?

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$$



shear

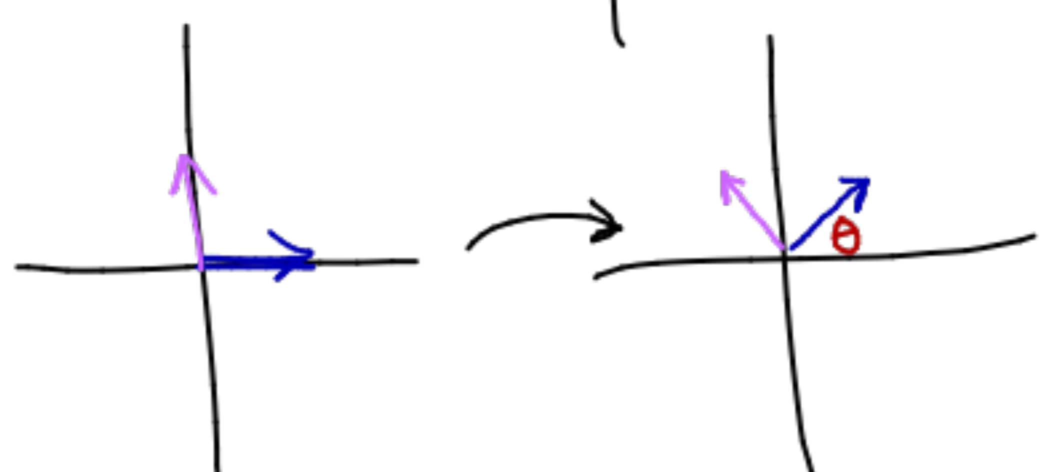
$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$$



rotate by $\pi/4$ and scale by $\sqrt{2}$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$



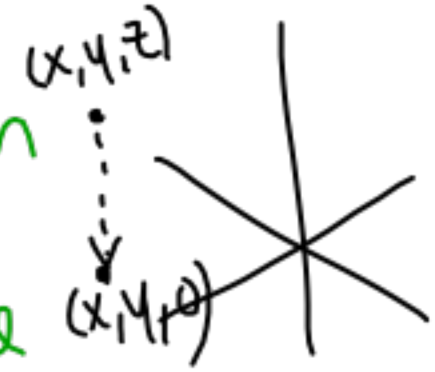
rotate by θ

Examples in $\mathbb{R}^3$

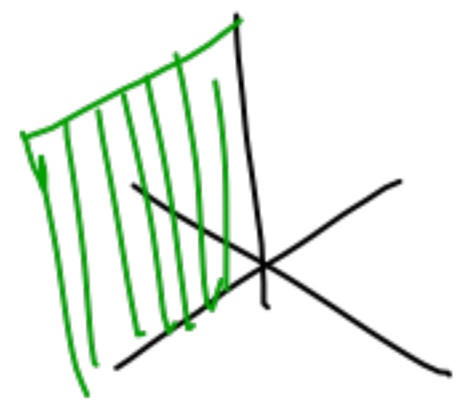
What do these matrices do to $\mathbb{R}^3$?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

projection to xy plane

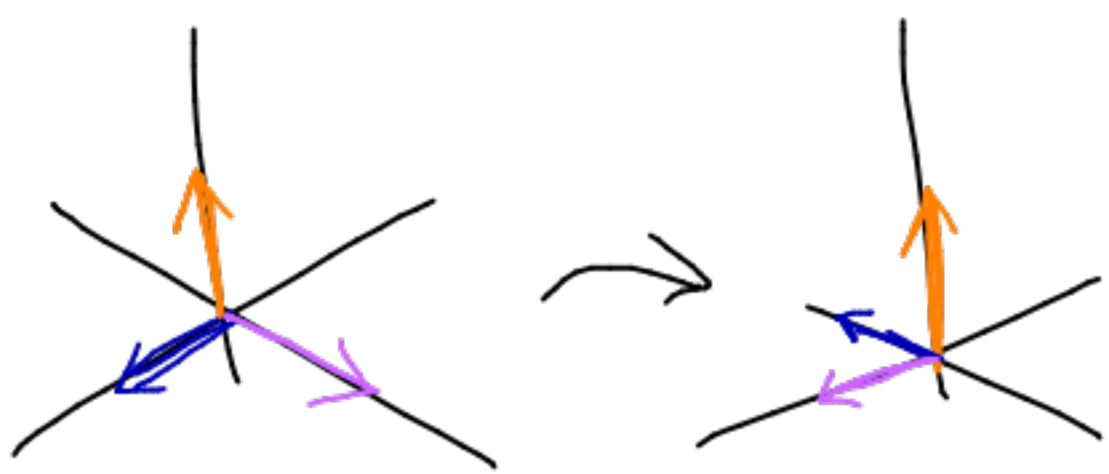


$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ z \end{pmatrix}$$



refl. in xz-plane

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



rotation about z-axis by $\pi/2$.

Linear Transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if

- $T(u + v) =$ $T(u)+T(v)$ all u, v in $\mathbb{R}^n$
- $T(cv) =$ $cT(v)$ all v in $\mathbb{R}^n$
  c in $\mathbb{R}$

If T is a linear function then

$$T(c_1 v_1 + \cdots + c_k v_k) = c_1 T(v_1) + \cdots + c_k T(v_k)$$

In engineering this is called superposition

Fact. Every matrix transformation T_A is linear.

Next time: Every linear function $\mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation.