Announcements Feb 8

- Please complete mid-semester CIOS evaluations this week
- WebWork 1.7 and 1.8 due Thursday
- WebWork 1.9 extra credit, due Thursday
- Midterm 1 in class this week Friday Feb 12 on Chapter 1
- Office Hours Tuesday and Wednesday 2-3, after class, and by appointment in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Baishen Wed 4-5, Matt Thu 3-4, Shivang Fri 10:30-11 + 12:30-1
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 1.9

The Matrix of a Linear Transformation

Linear transformations are matrix transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if

- $T(u+v) =$ $T(u)+T(v)$ for any u, v in $\mathbb{R}^n$
- $T(cv) =$ $cT(v)$ any v in $\mathbb{R}^n$, c in $\mathbb{R}$

Main point: If we know $T(e_1), \ldots, T(e_n)$, then we know every $T(v)$.

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ linear

$T(e_1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $T(e_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$e_1 = (1, 0, 0, \ldots, 0)$$
$$e_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$

Then $T\begin{pmatrix} 7 \\ 4 \end{pmatrix} = T(7e_1 + 4e_2) \overset{\text{linear}}{=} 7T(e_1) + 4T(e_2)$

$= 7\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 \\ 25 \end{pmatrix}$

Recall that every matrix transformation T_A is linear.

$T_A(v) = \underset{m\times n}{A}\,\underset{n\times 1}{v} = m\times 1$

We saw: $T_A(u+v) = A(u+v) = Au + Av = T_A(u) + T_A(v)$

Theorem. A function $\mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if it is a matrix transformation.

Linear transformations are matrix transformations

Theorem. A function $\mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if it is a matrix transformation.

This means that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$... there is an mxn matrix A so:

$$T_A(v) = Av = T(v)$$

Works because:

Why?

$$A = {}_m\begin{pmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{pmatrix}_n$$

$$T_A(v) = Av = \begin{pmatrix} T(e_1) & \cdots & T(e_n) \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

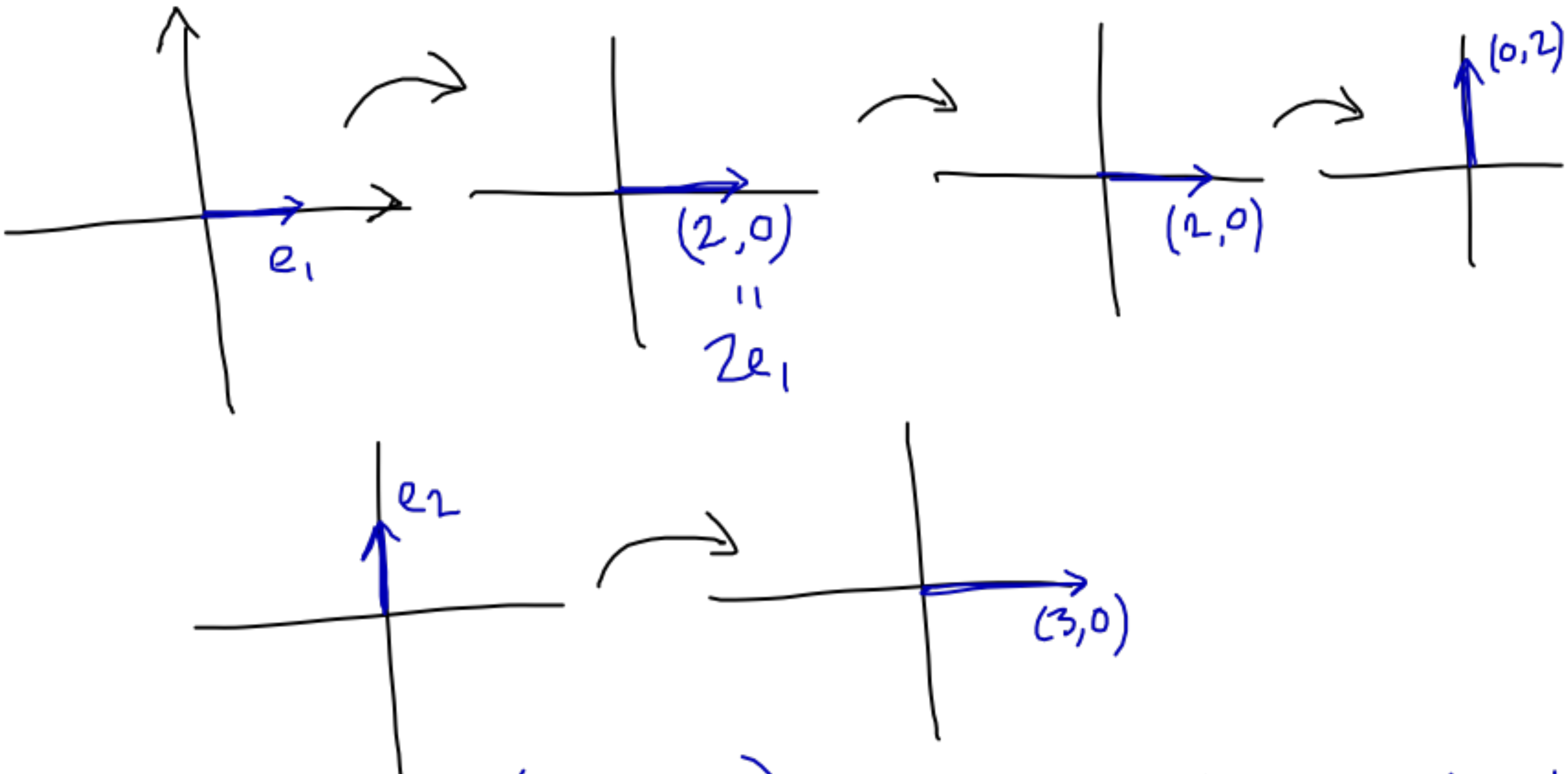
$$= v_1 T(e_1) + \cdots + v_n T(e_n)$$

$$= T(v_1 e_1 + \cdots + v_n e_n)$$

$$= T(v)$$

Linear transformations are matrix transformations

Q. Find the matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the x-direction and 3 in the y-direction, and then reflects over the line $y = x$.



$$A = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \qquad T\begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \end{pmatrix}$$

Linear transformations are matrix transformations

Q. Find the matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the y-axis and then rotates counterclockwise by $\pi/2$.

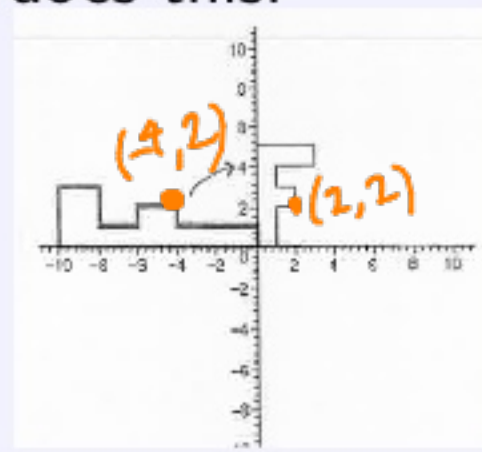
Linear transformations are matrix transformations

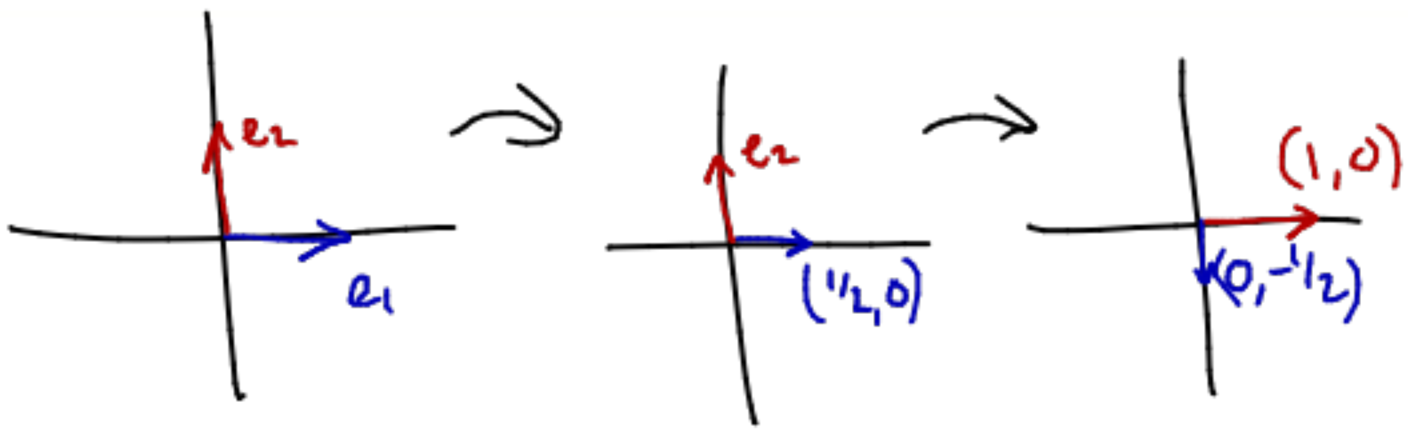
Q. Find the matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane.

Discussion

Discussion Question

Find a matrix that does this.





$$A = \begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

One-to-one

$T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if each b in $\mathbb{R}^m$ is the image of at most one v in $\mathbb{R}^n$.

Not one-to-one means two inputs w/ same output.

Theorem. Suppose $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation corresponding to a matrix A. Then the following are all equivalent:

- T_A is one-to-one
- the columns of A are... lin ind.
- $Ax = 0$ has... only 0 soln. $T_A(x)=0$
- A has a pivot... every col.

Q. What can we say about the relative sizes of m and n if T_A is one-to-one?

$m \geq n$.

Q. Draw a picture of the image of a one-to-one mapping $\mathbb{R} \to \mathbb{R}^3$

Onto

$T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if the image of T is $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the image of at least one v in $\mathbb{R}^m$.

Theorem. Suppose $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation corresponding to a matrix A. Then the following are all equivalent:

- T_A is onto
- the columns of A... span $\mathbb{R}^m$
- A has a pivot... in every row
- $Ax = b$ is consistent... for all b.

$Ax = T_A(x)$

Q. What can we say about the relative sizes of m and n if T_A is onto?

$n \geq m$

Q. Give an example of an onto mapping $\mathbb{R}^3 \to \mathbb{R}$

One-to-one and Onto

Do the following give linear transformations that are one-to-one? onto?

$$\begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$