Announcements Feb 8

• Please complete mid-semester CIOS evaluations this week

• WebWork 1.7 and 1.8 due Thursday

• WebWork 1.9 extra credit, due Thursday

• Midterm 1 in class this week Friday Feb 12 on Chapter 1

• Office Hours Tuesday and Wednesday 2-3, after class, and by appointment in Skiles 244 or 236

• LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Baishen Wed 4-5, Matt Thu 3-4, Shivang Fri 10:30-11 + 12:30-1

• Math Lab, Clough 280
  • Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  • Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  • LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6
Section 1.9

The Matrix of a Linear Transformation
Linear transformations are matrix transformations

A function \( T : \mathbb{R}^n \to \mathbb{R}^m \) is linear if

- \( T(u + v) = T(u) + T(v) \) for any \( u, v \) in \( \mathbb{R}^n \)
- \( T(cv) = cT(v) \) for any \( v \) in \( \mathbb{R}^n \), \( c \) in \( \mathbb{R} \)

Main point: if we know \( T(e_1), \ldots, T(e_n) \), then we know every \( T(v) \).

\[
e_1 = (1, 0, 0, \ldots, 0) \\
e_2 = (0, 1, 0, \ldots, 0) \\
 \vdots
\]

Recall that every matrix transformation \( T_A \) is linear.

\[
T_A(v) = AV = m \times n \\
A \times n \times n \times 1
\]

**Theorem.** A function \( \mathbb{R}^n \to \mathbb{R}^m \) is linear if and only if it is a matrix transformation.
Linear transformations are matrix transformations

**Theorem.** A function $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if and only if it is a matrix transformation.

This means that for any linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ there is an $m \times n$ matrix $A$ so:

$$T_A(v) = Av = T(v)$$

**Why?**

$$A = \left( T(e_1) \ T(e_2) \ \ldots \ T(e_n) \right)_{n \times m}$$

$T_A(v) = Av = (T(e_1) \ T(e_2) \ \ldots \ T(e_m))v = v_1 T(e_1) + \ldots + v_n T(e_n) = T(v_1 e_1 + \ldots + v_n e_n) = T(v)$
Linear transformations are matrix transformations

Q. Find the matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the $x$-direction and 3 in the $y$-direction, and then reflects over the line $y = x$.

$$A = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \quad T(\begin{pmatrix} 5 \\ -1 \end{pmatrix}) = \begin{pmatrix} 6 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \end{pmatrix}$$
Linear transformations are matrix transformations

Q. Find the matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the $y$-axis and then rotates counterclockwise by $\pi/2$. 
Linear transformations are matrix transformations

Q. Find the matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane.
Discussion

Discussion Question

Find a matrix that does this.

$$\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$
One-to-one

\( T : \mathbb{R}^n \to \mathbb{R}^m \) is **one-to-one** if each \( b \) in \( \mathbb{R}^m \) is the image of at most one \( v \) in \( \mathbb{R}^n \).

**Not one-to-one** means two inputs w/ same output.

**Theorem.** Suppose \( T_A : \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation corresponding to a matrix \( A \). Then the following are all equivalent:

- \( T_A \) is one-to-one
- the columns of \( A \) are lin ind.
- \( Ax = 0 \) has only 0 soln.
- \( A \) has a pivot... every col.

Q. What can we say about the relative sizes of \( m \) and \( n \) if \( T_A \) is one-to-one?

\[
\begin{array}{ccc}
\begin{array}{ccc}
\bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc \\
\end{array}
\end{array}
\]

\( M \geq N \)

Q. Draw a picture of the image of a one-to-one mapping \( \mathbb{R} \to \mathbb{R}^3 \)
Onto

\( T : \mathbb{R}^n \to \mathbb{R}^m \) is onto if the image of \( T \) is \( \mathbb{R}^m \), that is, each \( b \) in \( \mathbb{R}^m \) is the image of at least one \( v \) in \( \mathbb{R}^m \).

**Theorem.** Suppose \( T_A : \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation corresponding to a matrix \( A \). Then the following are all equivalent:

- \( T_A \) is onto
- the columns of \( A \) span \( \mathbb{R}^m \)
- \( A \) has a pivot in every row
- \( Ax = b \) is consistent for all \( b \).

**Q.** What can we say about the relative sizes of \( m \) and \( n \) if \( T_A \) is onto?

\[ n \geq m \]  
(\( \square \)(\( \square \))

**Q.** Give an example of an onto mapping \( \mathbb{R}^3 \to \mathbb{R} \)
One-to-one and Onto

Do the following give linear transformations that are one-to-one? onto?

\[
\begin{pmatrix}
1 & 0 & 7 \\
0 & 1 & 2 \\
0 & 0 & 9
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 1 & 1
\end{pmatrix}
\]