Section 1.2

Row Reduction and Echelon Forms

Row Reduction and Echelon Forms

A matrix is in row echelon form if

1. all zero rows are at bottom
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row
3. below a leading entry of a row, all entries are zero

can be 0

$\star$ numbers

$$\left(\begin{array}{cccc|c} \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

pivots $\neq 0$

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 6 \end{array}\right)$$

This system is easy to solve using back substitution.

The pivot positions are the leading (nonzero) entries in each row.

Reduced Row Echelon Form

A system is in *reduced* row echelon form if also:

4. the ~~leading entry~~ pivot in each nonzero row is 1
5. each ~~leading entry~~ pivot of a row is only nonzero entry in that col.

For example:

$$\begin{array}{cc|c} x & y & \\ \hline \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 6 \end{array}\right) \qquad \begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This system is even easier to solve.

Can every matrix be put in reduced row echelon form?

Reduced Row Echelon Form

Poll

Which are in reduced row echelon form?

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Row Reduction

Theorem. Each matrix is equivalent to one and only one matrix in reduced row echelon form.

We'll give an algorithm. That shows a matrix is equivalent to at least one matrix in reduced row echelon form.

Row Reduction Algorithm

Step 1a Swap the 1st row with a lower one so a leftmost nonzero entry is in 1st row

Step 1b Scale 1st row so that its leading entry is equal to 1

Step 1c Use row replacement so all entries above and below this 1 are 0

Step 2a Cover the first row, swap the 2nd row with a lower one so that the leftmost nonzero (uncovered) entry is in 2nd row; uncover 1st row

etc.

Example. $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{pmatrix}$

$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

More examples

Solve the following systems.

$$(i)\quad \begin{aligned} 3x + y + 3z &= 2 \\ x + 2z &= -3 \\ 2x + y + z &= 4 \end{aligned} \qquad (ii)\quad \begin{aligned} a + 2b + d &= 3 \\ c + d - 2e &= 1 \end{aligned}$$

Solutions of Linear Systems

We want to go from reduced row echelon forms to solutions of linear systems.

Solve the linear system associated to:

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

What are the solutions?

$x = 5$
$y = 2$

Geometrically, the solution is a pt


Park
This way to:
The Promenade
Legacy Fountain
DOGS PROHIBITED ON FOUNTAIN DECK

Solutions of Linear Systems: Free Variables I

We want to go from reduced row echelon forms to solutions of linear systems.

Solve the linear system associated to:

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

not a pivot

This represents two equations:

$$x_1 + 5x_3 = 0$$
$$x_2 + 2x_3 = 1$$

What are the solutions?

$X_1 = -5X_3$

$X_2 = 1 - 2X_3$

$X_3 =$ ~~anything~~ free

some solns:
(0,1,0)
(-5,-1,1)

Geometrically, the solution is a line.

In general the free variables correspond to non-pivots

Solutions of Linear Systems: Free Variables II

Solve the linear system associated to:

$$\left(\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$x_1 + 5x_3 = 0$

$x_4 = 0$

$x_1 = -5x_3$

$x_2 = \text{free}$

$x_3 = \text{free}$

$x_4 = 0$

a soln

$(0, 0, 0, 0)$

$(-10, 1, 2, 0)$

plane.

Solutions of Linear Systems: Consistency

Solve the linear system associated to:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The second row gives

$0 = 1$

inconsis.

Consistent Systems

Theorem

A linear system is consistent if and only if (exactly when) the last column of the augmented matrix does not have a pivot. If it is consistent, the solution can be a point, line, plane, etc.

Poll

A linear system has 4 variables and 3 equations. What are the possible solution sets?

1. nothing
2. point
3. line
4. plane
5. 3-dimensional plane
6. 4-dimensional plane