Section 1.4

The Matrix Equation $Ax = b$

Multiplying Matrices

row $\times$ column : $\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1b_1 + a_2b_2 + \cdots a_nb_n$

length n vector

matrix $\times$ column : $\begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$ column vector

same

OR

matrix $\times$ column : $\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = b_1c_1 + \cdots + b_nc_n$

Example:

row way column way

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 5\cdot 2 + 6\cdot 3 \\ 7\cdot 2 + 8\cdot 3 \end{pmatrix}}_{\begin{pmatrix} 28 \\ 38 \end{pmatrix}} = 2\begin{pmatrix} 5 \\ 7 \end{pmatrix} + 3\begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

Linear Systems vs Matrix Equations vs Vector Equations

$$\begin{aligned} 2x + 3y &= 7 \\ x - y &= 5 \end{aligned}$$

linear system

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

matrix eqn

$$\begin{aligned} 2x + 3y &= 7 \\ 1\cdot x - 1\cdot y &= 5 \end{aligned}$$

$$\begin{array}{cc} x & y & \\ \end{array}$$
$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$$

vector eqn

$$x\begin{pmatrix} 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

We will go back and forth between matrix equations and vector equations over and over again.

Matrix Equations vs Vector Equations

Q. Say $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

Write $3u - 5v + 7w = 0$ as a matrix equation.

$$\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} u & v & w \end{pmatrix}$$

Solutions to Linear Systems vs Spans

Fact. $Ax = b$ has a solution $\Longleftrightarrow$ b is in the span of columns of A.

$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$

$b = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

Why?

$Av = b$ for some v

$\Updownarrow$

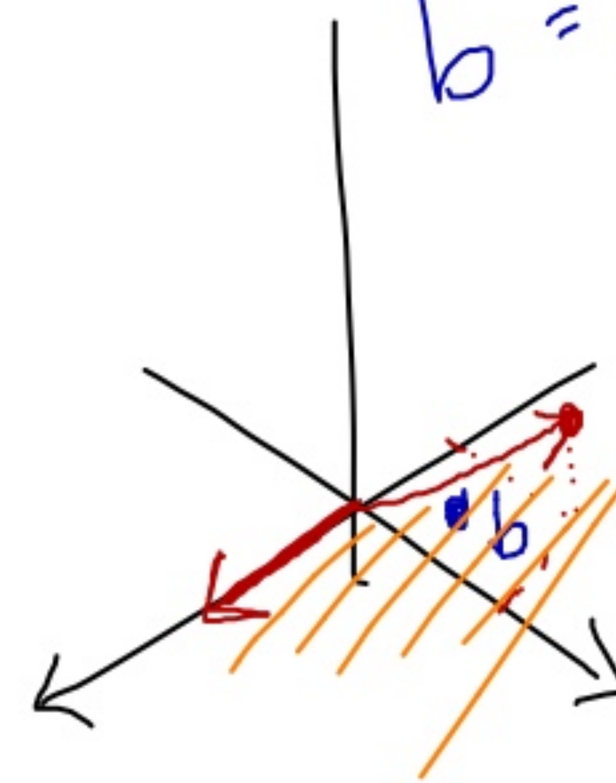
$\begin{pmatrix} c_1 & c_2 & \cdots & c_m \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

cols of A

$\|$

$v_1c_1 + v_2c_2 + \cdots + v_mc_m = b$

lin combo of cols of A



$Ax = b$

soln: $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Again this is a basic fact we will use over and over.

Is a given vector in the span?

Q. Which of the following vectors are in the span of

$$(2,3,1,4,0), (3,4,-1,3,5), (1,-1,2,4,3)?$$

- $(3,6,-5,-2,-7)$
- $(6,19,-3,4,-12)$

$$a\begin{pmatrix}2\\3\\1\\4\\0\end{pmatrix}+b\begin{pmatrix}3\\4\\-1\\3\\5\end{pmatrix}+c\begin{pmatrix}1\\-1\\2\\4\\3\end{pmatrix}=\begin{pmatrix}3\\6\\-5\\-2\\-7\end{pmatrix}$$

$$\begin{pmatrix}2&3&1\\3&4&-1\\1&-1&2\\4&3&4\\0&5&3\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}3\\6\\-5\\-2\\-7\end{pmatrix}$$

$\rightsquigarrow$ row reduce the aug. mat.

Is a given vector in the span?

Poll

Which of the following true statements can be checked without row reduction?

1. $(0, 1, 2)$ is in the span of $(3, 3, 4)$, $(0, 10, 20)$, $(0, -1, -2)$
2. $(0, 1, 2)$ is in the span of $(3, 3, 4)$, $(0, 5, 7)$, $(0, 6, 8)$
3. $(0, 1, 2)$ is in the span of $(3, 3, 4)$, $(0, 1, 0)$, $(0, 0, \sqrt{2})$
4. $(0, 1, 2)$ is in the span of $(5, 7, 0)$, $(6, 8, 0)$, $(3, 3, 4)$

Pivots vs Solutions

Theorem. Let A be an $m \times n$ matrix. The following are equivalent.

1. $Ax = b$ has a solution for all b in $\mathbb{R}^m$
2. The span of the columns of A is $\mathbb{R}^m$
3. A has a pivot in each row

Why?

$$\begin{array}{c} x_1 \cdots x_n \\ m\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \\ m \times n \end{array} \begin{array}{c} \\ \begin{pmatrix} \\ \\ \\ \end{pmatrix} \\ n \times 1 \end{array} = \begin{array}{c} \begin{pmatrix} \\ \\ \\ \end{pmatrix} \\ m \times 1 \end{array}$$

examples: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = \begin{pmatrix} 17 \\ 32 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix} x = \begin{pmatrix} 17 \\ 32 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} \square & * & * & * \\ 0 & \square & * & * \\ 0 & 0 & 0 & 17 \end{array}\right)$$

Properties of the Matrix Product Ax

$c =$ real number, $u, v =$ vectors,

- $A(u + v) =$ $Au + Av$
- $A(cv) =$ cAv

$A(3u - 7v)$
$= 3Au - 7Av$

Application. If u and v are solutions to $Ax = 0$ then so is ev~~ery element of~~ ~~Span$\{u, v\}$.~~ each vector in Span$\{u,v\}$

$Au = 0$

$Av = 0$

$A(u+v) = Au + Av = 0 + 0 = 0$

$A(u-v) = 0$

Solutions to $Ax = b$

Poll

If $b \neq 0$ then the set of solutions to $Ax = b$ is

1. always a span
2. sometimes a span
3. never a span