Announcements Jan 29

- Announcing: extra credit (bounty) for WebWork bugs: post a screen shot and a clear explanation
- Written Homework 2 due now. Circle H/J and L/C/R. Pass to aisle, then front.
- Quiz 2 in class Today on Sections 1.3-1.4.
- Midterm 1 in class Friday Feb 12
- My Office Hours Tuesday and Wednesday 2-3, after class, and by appointment
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Baishen Wed 4-5, Matt Thu 3-4, Shivang Fri 10:30-11 + 12:30-1
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12.
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 1.5

Solution Sets of Linear Systems

Homogeneous systems

Homogeneous systems $\longleftrightarrow$ matrix equations $Ax = 0$

Homogenous systems always have the trivial solution: $x = 0$

$Ax = 0$ has a nonzero solution $\Leftrightarrow$ there is a free variable. $\Leftrightarrow$ row with no pivot.

Homogeneous systems
Example

How many free variables for $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

Homogeneous systems

Example

How many free variables for $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2 free vars

solns: plane.

Homogeneous systems
Example

How many free variables for $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

1 free var

solns: line.

How many free variables for $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

3 free vars

solns: 3-dim plane in $\mathbb{R}^4$

Dimension and Span of Homogeous Systems

- If $v_1, \ldots, v_k$ are solutions to $Ax = 0$, then so is... $\mathrm{Span}\{v_1, \ldots, v_k\}$

  Why? $A(5v_1 - 7v_3)$
  $= 5Av_1 - 7Av_3$
  $= 0 - 0 = 0$

- $\leadsto$ set of solutions to $Ax = 0$ is... line, plane, ...

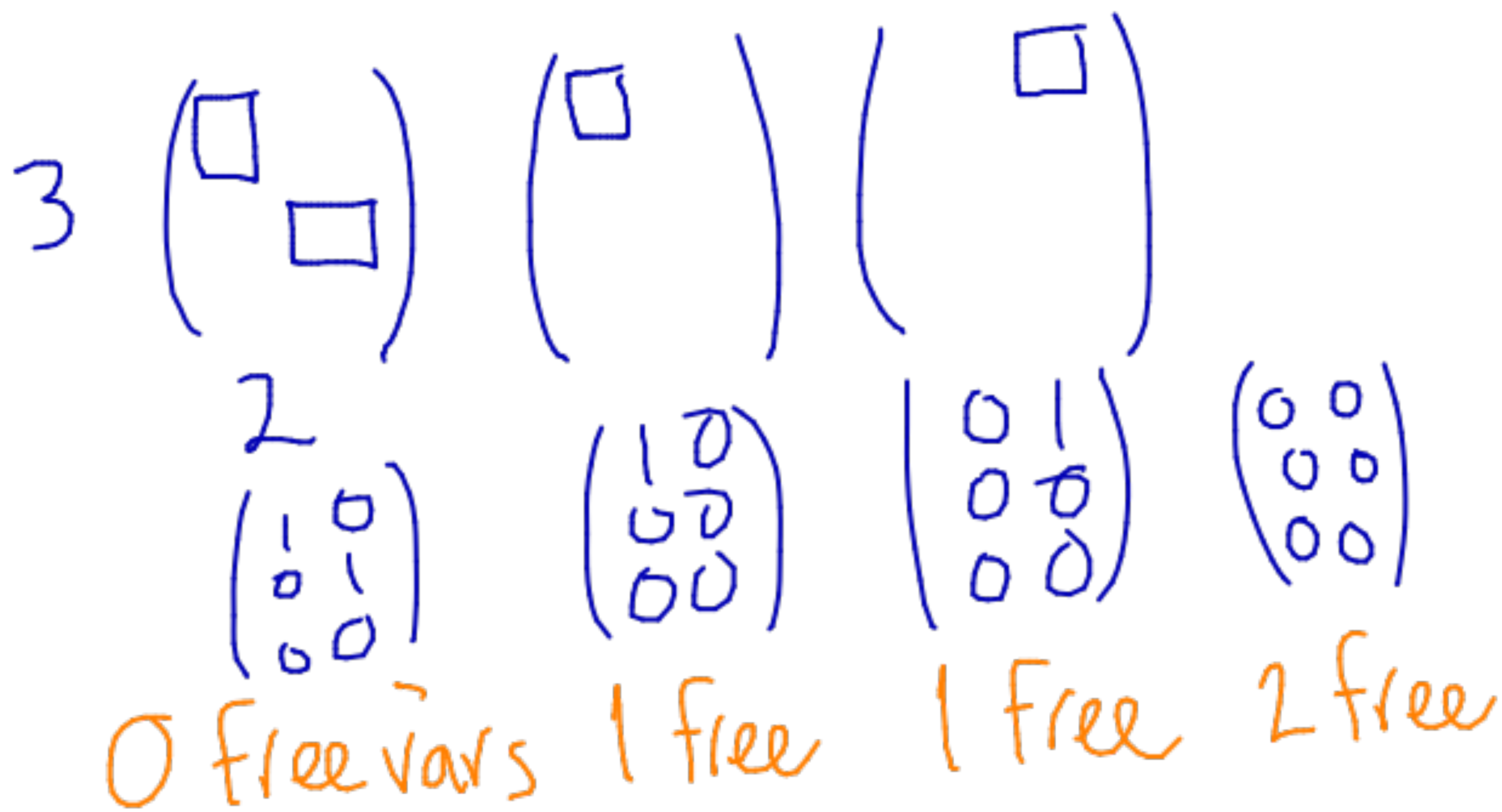
Variables, equations, and dimension

Poll

How many solutions can there be for a homogeneous system with more equations than variables?

1. 0
2. 1
3. ∞

What about more variables than equations?



Parametric Forms

Say free variables for $Ax = 0$ are $x_k, \ldots, x_n$.

Then the solutions to $Ax = 0$ can be written as

$$x_k v_k + x_{k+1} v_{k+1} + \cdots + x_n v_n$$

for some $v_k, \ldots, v_n$ (in other words, as a span of the v's).

This is the *parametric form* of the solutions.

Parametric Forms for Solutions

Homogeneous case

Find the parametric solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

Parametric Forms for Solutions

Homogeneous case

Find the parametric solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ -2 & -3 & 4 & 5 & 0 \\ 2 & 4 & 0 & -2 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & -8 & -7 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - 8x_3 - 7x_4 = 0$

$x_1 = 8x_3 + 7x_4$

$x_2 = -3x_4 - 4x_3$

$x_3 = \cancel{\text{free}}\ 1 \cdot x_3$

$x_4 = \cancel{\text{free}}\ 1 \cdot x_4$

param form solns: $x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$

$\text{Span}\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

Note: don't really need the last column!

Parametric Forms for Solutions

Homogeneous case

Find the parametric solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

Parametric Forms for Solutions

Homogeneous case

Find the parametric solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$x_1 = -x_4$
$x_2 = x_4$
$x_3 = -x_4$
$x_4 = x_4$

$$x_4 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

Parametric Forms for Solutions

Homogeneous case

Find the parametric solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

$$x_1 = -x_2 - x_3 - x_4$$

$$x_2 = x_2$$

$$x_3 = x_3$$

$$x_4 = x_4$$

$$x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Nonhomogeneous Systems

Suppose $Ax = b$, and $b \neq 0$.

As before, we can find the parametric solution in terms of free variables.

What is the difference?

Parametric Forms for Solutions

Nonhomogeneous case

Find the parametric solution to $Ax = (3, 2, 6)$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

Parametric Forms for Solutions

Nonhomogeneous case

Find the parametric solution to $Ax = (3, 2, 6)$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 3 \\ -2 & -3 & 4 & 5 & 2 \\ 2 & 4 & 0 & -2 & 6 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & -8 & -7 & -13 \\ 0 & 1 & 4 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

soln to $Ax=0$

$$x_1 = -13 + 8x_3 + 7x_4$$
$$x_2 = 8 - 4x_3 - 3x_4$$
$$x_3 = \cancel{\text{free}}\ x_3$$
$$x_4 = \cancel{\text{free}}\ x_4$$

param

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -13 \\ 8 \\ 0 \\ 0 \end{pmatrix}$$

only new thing

$$\begin{pmatrix} -13 \\ 8 \\ 0 \\ 0 \end{pmatrix} + \text{span}\left\{ \begin{pmatrix} \; \end{pmatrix}, \begin{pmatrix} \; \end{pmatrix} \right\}$$

Parametric Forms for Solutions

Nonhomogeneous case

Find the parametric solution to $Ax = (4, 2, 4)$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

Parametric Forms for Solutions

Nonhomogeneous case

Find the parametric solution to $Ax = (4, 2, 4)$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$$

$x_1 = 2 - x_4$
$x_2 = x_4$
$x_3 = 2 - x_4$
$x_4 = \text{free}$

$$x_4 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

Parametric Forms for Solutions

Nonhomogeneous case

Find the parametric solution to $Ax = (9)$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 9 \end{array}\right)$$

Homogeneous vs. Nonhomogeneous Systems

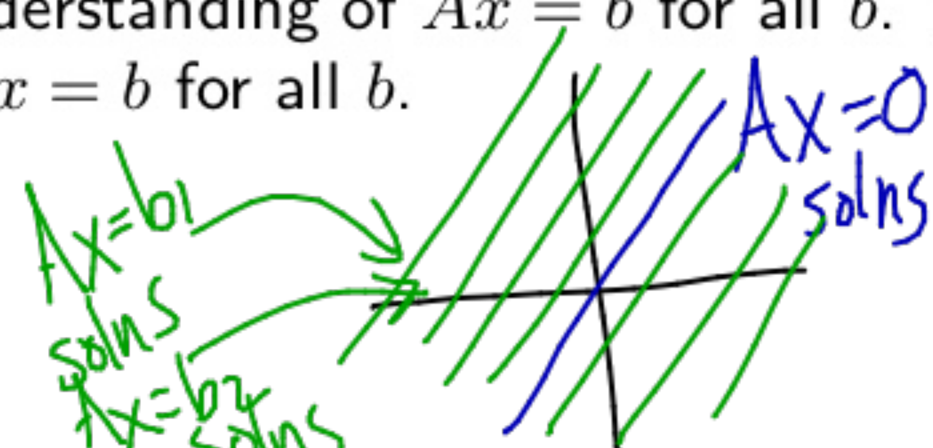
Key realization. Set of solutions to $Ax = b$ obtained by taking one solution and adding all possible solutions to $Ax = 0$.

$$Ax = 0 \text{ solutions} \rightsquigarrow Ax = b \text{ solutions}$$

$$x_k v_k + \cdots + x_n v_n \rightsquigarrow \quad p + x_k v_k + \cdots + x_n v_n$$

So: set of solutions to $Ax = b$ is parallel to the set of solutions to $Ax = 0$.

So by understanding $Ax = 0$ we gain understanding of $Ax = b$ for all b. This gives structure to the set of equations $Ax = b$ for all b.



Homogeneous vs. Nonhomogeneous Systems

Varying b

What are the solutions to $Ax = b$ for various b where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}?$$

$$Ax = 0: \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$