Announcements Mar 14

- WebWork 5.1 due Thursday (not up yet)
- Quiz 7 on 5.1 on Friday
- Homework 7 to be completed on Piazza
- Midterm 3 in class Friday April 8 on Chapter 5
- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6
Chapter 5
Eigenvalues and Eigenvectors
Section 5.1

Eigenvalues and Eigenvectors
Where are we?

Remember:

Almost every engineering problem, no matter how huge, can be reduced to linear algebra:

\[ Ax = b \quad \text{or} \quad Ax = \lambda x \]

We have said most of what we are going to say about the first problem. We now begin in earnest on the second problem.
A Question from Biology

In a population of rabbits...
- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population one year, what is the population the next year?

\[
A = \begin{pmatrix}
0 & 6 & 8 \\
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0
\end{pmatrix}
\begin{pmatrix}
f_0 \\
S_0 \\
t_0
\end{pmatrix} = 
\begin{pmatrix}
f_1 \\
S_1 \\
t_1
\end{pmatrix}
\]

Now choose some starting population vector \( u = (f, s, t) \) and choose some number of years \( N \). What is the new population after \( N \) years? \( N + 1 \) years?

\[
A \cdots AA v_0 = A^{10} v_0 \\
& A^{20} v
\]

Use a computer to find the actual numbers.
A Question from Biology

<table>
<thead>
<tr>
<th>Year 0</th>
<th>Year 10</th>
<th>Year 11</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>30189</td>
<td>61316</td>
</tr>
<tr>
<td>7/9</td>
<td>7761</td>
<td>15095</td>
</tr>
<tr>
<td>1/3</td>
<td>1844</td>
<td>3881</td>
</tr>
<tr>
<td></td>
<td>9459</td>
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<tr>
<td>4</td>
<td>7405</td>
<td>14428</td>
</tr>
<tr>
<td>1/8</td>
<td>1765</td>
<td>3703</td>
</tr>
</tbody>
</table>

Observation 1: All 3 segments of the population double each year.
Observation 2: Ratios get close to 16:4:1.

2 is an eigenvalue.
Eigenvectors and Eigenvalues

Suppose $A$ is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and $\lambda \in \mathbb{R}$ so that

$$Av = \lambda v$$

then $v$ is called an eigenvector for $A$, and $\lambda$ is the corresponding eigenvalue.

This the the most important definition in the course.
Eigenvalues and Eigenvalues

Examples

\[ A \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix}, \quad \lambda = 2 \]

\[
\begin{pmatrix} 64 \\ 16 \\ 4 \end{pmatrix}
\]

\[ A = \begin{pmatrix} 2 & -4 \\ 2 & 8 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda = 4 \]

How do you check?
Eigenvectors and Eigenvalues

Confirming eigenvectors

Which of \(
\begin{pmatrix}
1 \\ 1
\end{pmatrix},
\begin{pmatrix}
1 \\ -1
\end{pmatrix},
\begin{pmatrix}
-1 \\ 1
\end{pmatrix},
\begin{pmatrix}
2 \\ 1
\end{pmatrix},
\begin{pmatrix}
0 \\ 0
\end{pmatrix}
\) are eigenvectors of

\[
\begin{pmatrix}
1 & 1 \\ 1 & 1
\end{pmatrix}
\]?

What are the eigenvalues?

\[
\begin{align*}
\begin{pmatrix}
1 & 1 \\ 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\ 1
\end{pmatrix} &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{yes } \lambda = 2 \\
\begin{pmatrix}
1 & 1 \\ 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\ -1
\end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{yes } \lambda = 0. \\
\begin{pmatrix}
1 & 1 \\ 1 & 1
\end{pmatrix}
\begin{pmatrix}
-1 \\ 1
\end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{yes } \lambda = 0. \\
\begin{pmatrix}
1 & 1 \\ 1 & 1
\end{pmatrix}
\begin{pmatrix}
2 \\ 1
\end{pmatrix} &= \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \text{no}
\end{align*}
\]

\[
\begin{pmatrix}
1 & 1 \\ 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\ 0
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{no because 0-vector is never an eigenvector}
\]
Eigenvectors and Eigenvalues

Confirming eigenvalues

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$. 

\[
\begin{pmatrix} \frac{2}{-1} & -4 \\ -1 & \frac{-1}{-1} \end{pmatrix} (-4) = \begin{pmatrix} \frac{-12}{3} \\ \frac{20}{60} = \frac{60}{-15} \end{pmatrix}
\]

\[A_v - 3v = 0 \quad \text{row reduce!}
\]

\[\begin{pmatrix} 2 - 4 \\ -1 - 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}
\]

\[-x_1 - 4x_2 = 0
\]

\[x_2 \begin{pmatrix} -4 \\ 1 \end{pmatrix} \quad \text{a whole line of eigenvectors.}
\]

What is a general procedure for finding eigenvalues?
Eigenspaces

Let $A$ be an $n \times n$ matrix. The set of eigenvectors for a given eigenvalue $\lambda$ of $A$ is a subspace of $\mathbb{R}^n$ called the $\lambda$-eigenspace of $A$.

Why is this a subspace?

Fact. $\lambda$-eigenspace for $A = \text{Nul}(A - \lambda I)$

Example. Find the eigenvalues, eigenvectors, and eigenspaces and sketch.

\[
\begin{pmatrix}
5 & -6 \\
3 & -4
\end{pmatrix}
\]
Eigenspaces

Bases

Find a basis for the 2–eigenspace:

\[
A = \begin{pmatrix}
4 & -1 & 6 \\
2 & 1 & 6 \\
2 & -1 & 8
\end{pmatrix}
\]

\[A - 2I\]

Find \( \text{Nul} \left( \begin{pmatrix}
2 & -1 & 6 \\
2 & -1 & 6 \\
2 & -1 & 6
\end{pmatrix} \right) \)

\( \rightarrow \) 2–dim plane in \( \mathbb{R}^3 \).
Eigenvalues
Triangular matrices

**Fact.** The eigenvalues of a triangular matrix are the diagonal entries.

*Why?*
**Eigenvalues**

And invertibility

**Fact.** $A$ invertible $\iff 0$ is not an eigenvalue of $A$

Why?

\[
\begin{align*}
0 \text{ is an eigenvalue of } A & \iff (A-0 \cdot I)v = 0 \text{ has non-zero solns} \\
& \iff Av = 0 \text{ has non-0 solns} \\
& \iff A \text{ is not invertible.}
\end{align*}
\]
Eigenvalues

Distinct eigenvalues

**Fact.** If $v_1 \ldots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots \lambda_k$, then $\{v_1, \ldots, v_k\}$ are linearly independent.

*Why?* $k=2$

$\lambda=2$ eigenvector

$\lambda=3$ eigenvector
Eigenvectors and difference equations

Say we want to solve $x_{k+1} = Ax_k$. In other words, we need a sequence $x_0, x_1, x_2, \ldots$ with $x_1 = Ax_0$, $x_2 = Ax_1$, etc.

Example. $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a + b \end{pmatrix}$.
Say we have a column with a compressive force. How will the column buckle?

Approximate column with finite number of points, say

\[(0, 0), (0, 1), (0, 2), \ldots (0, 5), (0, 6)\]

Buckling leads to (roughly)

\[(0, 0), (x_1, 1), (x_2, 2), \ldots (x_5, 5), (0, 6)\]

Engineers use a difference equation to model this

\[x_{i-1} - 2x_i + x_{i+1} + \lambda x_i = 0\]
Difference equation for column buckling:

\[ x_{i-1} - 2x_i + x_{i+1} + \lambda x_i = 0 \]

We solve this difference equation as above. We need the eigenvector of

\[
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & 2 \\
\end{pmatrix}
\]

For most \( \lambda \), the only eigenvector is 0, which corresponds to no buckling. This matrix has three eigenvalues, 1, 2, and 3.