Announcements Mar 16

- WebWork 5.1 due Thursday
- Quiz 7 on 5.1 on Friday
- Homework 7 to be completed on Piazza
- Midterm 3 in class Friday April 8 on Chapter 5
- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

Eigenvectors and Eigenvalues

Where are we?

Remember:

> Almost every engineering problem, no matter how huge, can be reduced to linear algebra:
>
> $$Ax = b \quad \text{or}$$
> $$Ax = \lambda x$$

We have said most of what we are going to say about the first problem. We now begin in earnest on the second problem.

A Question from Biology

In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population one year, what is the population the next year?

$$A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ s_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} \boxed{f_1} \\ \boxed{s_1} \\ t_1 \end{pmatrix} \qquad A v_0 = v_1$$

Now choose some starting population vector $u = (f, s, t)$ and choose some number of years N. What is the new population after N years? $N + 1$ years?

$$A \cdots A A v_0 = A^{10} v_0 \qquad A^{20} v$$

$$\& \ A^{21} v$$

Use a computer to find the actual numbers.

A Question from Biology

Year 0	Year 10	Year 11
$\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 30189 \\ 7761 \\ 1844 \end{pmatrix}$	$\begin{pmatrix} 61316 \\ 15095 \\ 3881 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 9459 \\ 2434 \\ 577 \end{pmatrix}$	$\begin{pmatrix} 19222 \\ 4729 \\ 1217 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 28856 \\ 7405 \\ 1765 \end{pmatrix}$	$\begin{pmatrix} 58550 \\ 14428 \\ 3703 \end{pmatrix}$

Observation 1 All 3 segments of the popln double each year.

2 is an eigenvalue

Observation 2 Ratios get close to 16 : 4 : 1

eigen vector

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

eigen $=$ *characteristic*

This the the most important definition in the course.

Eigenvectors and Eigenvalues

Examples

$$\boxed{Av = \lambda v \qquad v \neq 0, \ \lambda \text{ in } \mathbb{R}}$$

$$A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix}, \quad \lambda = 2$$

$$\begin{pmatrix} 64 \\ 16 \\ 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda = 4$$

How do you check?

Eigenvectors and Eigenvalues

Confirming eigenvectors

$\boxed{Av = \lambda v \quad v \neq 0,\ \lambda \text{ in } \mathbb{R}}$

Which of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}?$$

What are the eigenvalues?

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ yes $\lambda = 2$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ yes $\lambda = 0.$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ yes $\lambda = 0.$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ no

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ no because 0-vect is never an e-vector

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 20 \\ -5 \end{pmatrix} = \begin{pmatrix} 60 \\ -15 \end{pmatrix}$

want $Av = 3v$

$Av - 3Iv = 0$

$Ax = 0$

$(A - 3I)v = 0$ row reduce!

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

$-x_1 - 4x_2 = 0$

$\rightsquigarrow x_2 \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ a whole line of eigenvectors.

$A - \lambda I$ not invertible.

What is a general procedures for finding eigenvalues?

Eigenspaces

Let A be an $n \times n$ matrix. The set of eigenvectors for a given eigenvalue λ of A is a subspace of $\mathbb{R}^n$ called the λ-eigenspace of A.

plus 0 vector.

Why is this a subspace?

$Av = \lambda v$

$(A-\lambda I)v = 0$

$Av = 7v$

$Aw = 7w$

$A(v+w) = Av + Aw$

$= 7v + 7w$

$= 7(v+w)$

Fact. λ-eigenspace for $A = \mathrm{Nul}(A - \lambda I)$

Example. Find the eigenspaces for $\lambda = 2$ and $\lambda = -1$ and sketch.

$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$

$A - 2I = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\underline{\lambda = 2}$

$\mathrm{Nul}\begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix}$

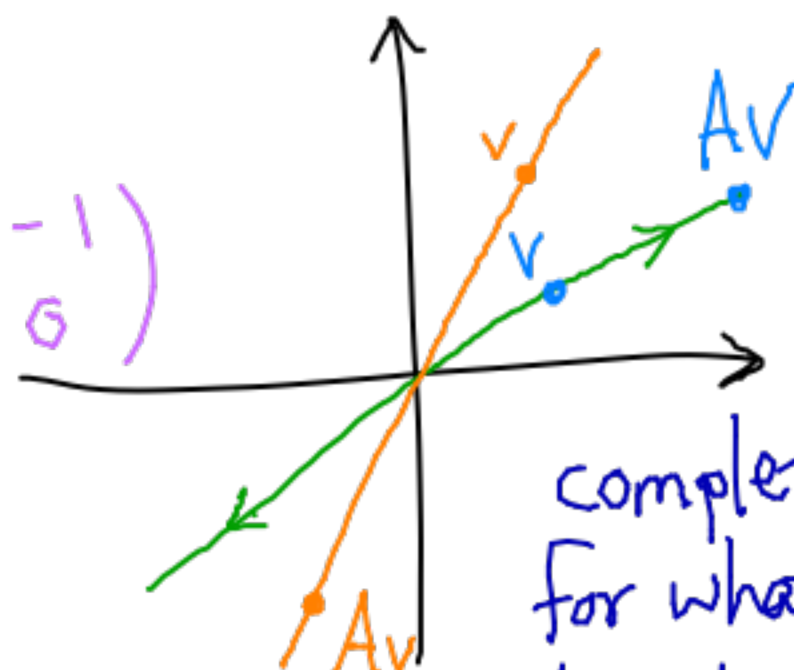
$= \mathrm{Nul}\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$

$x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{\lambda = -1}$

$\begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



complete pic for what A does to $\mathbb{R}^2$.

Find a basis for the 2–eigenspace:

λ

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$$

$A - 2I$

Find $\mathrm{Nul}\begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix}$

$\rightsquigarrow$ 2-dim plane in $\mathbb{R}^3$.

$$2x_1 - x_2 + 6x_3 = 0$$

basis

$$x_2\begin{pmatrix} \boxed{1/2} \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} \boxed{-3} \\ 0 \\ 1 \end{pmatrix}$$

Eigenvalues

And invertibility

Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A

Why?

0 is an eigenvalue of A

$\Longleftrightarrow$ $(\underline{A - 0 \cdot I})v = 0$ has non-zero solns

$\Longleftrightarrow$ $Av = 0$ has non-0 solns

$\overset{\text{IMT}}{\Longleftrightarrow}$ A not invertible.

upper or lower

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

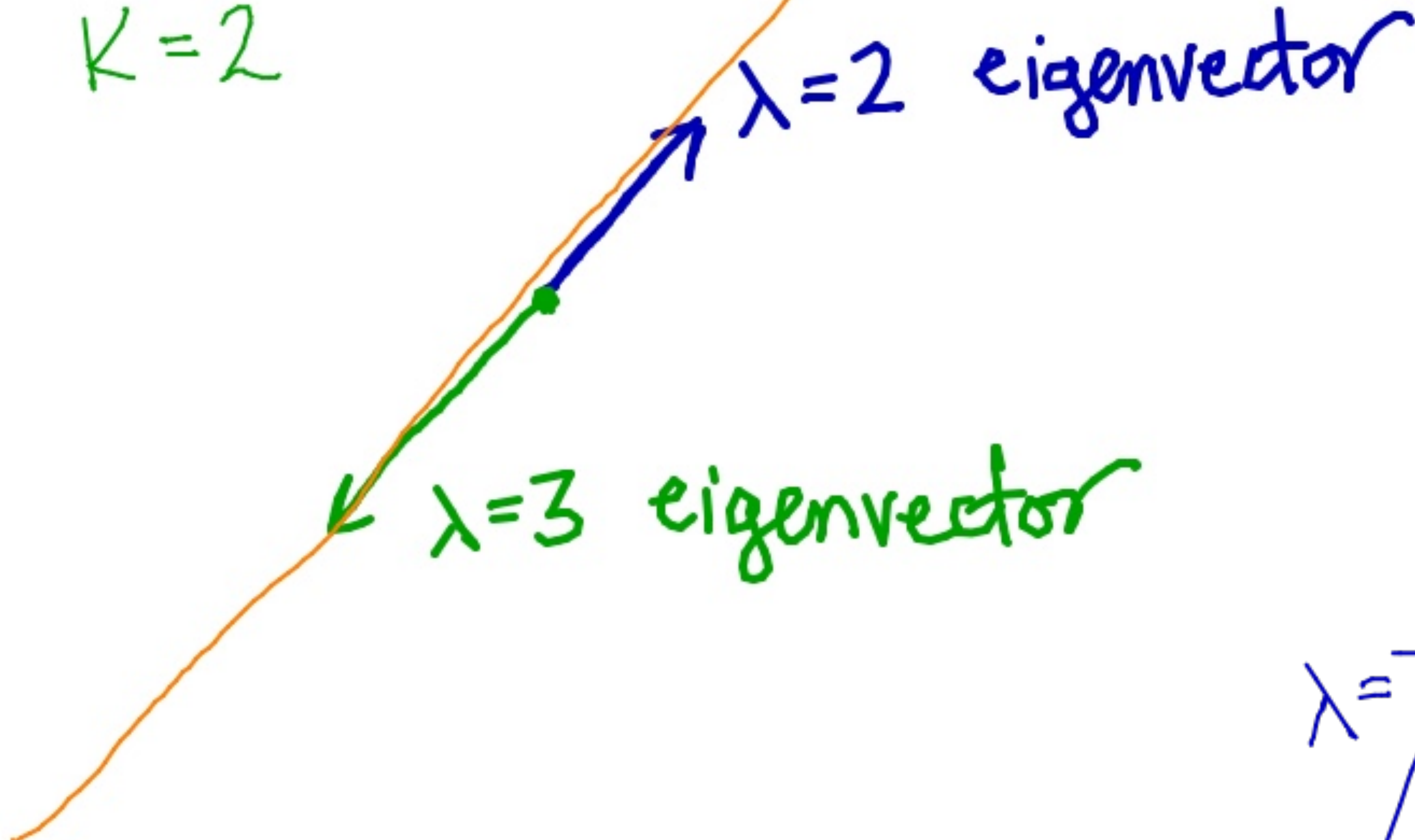
If you subtract 1, 4, or 6 from diagonal, you lose a pivot $\rightsquigarrow$ not invertible.

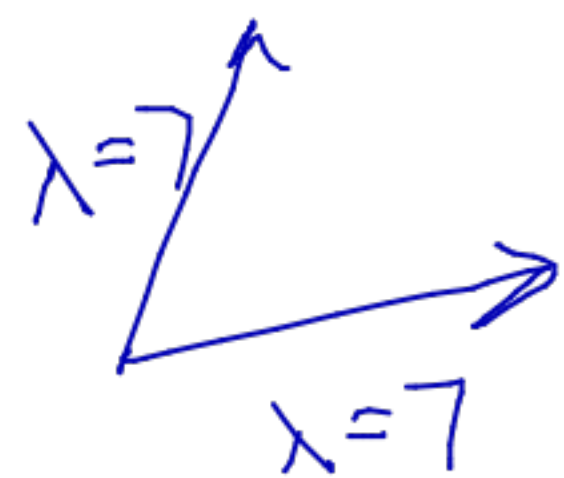
Eigenvalues

Distinct eigenvalues

Fact. If $v_1 \dots v_k$ are ~~distinct~~ eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.

Why? $K = 2$





Section 5.2

The characteristic polynomial

5.2 The characteristic polynomial

Outline

- The characteristic polynomial: a systematic way to find eigenvalues
  - 2×2 matrices
  - 3×3 matrices
- algebraic multiplicity of eigenvalues
- similar matrices $\rightsquigarrow$ same eigenvalues

Characteristic polynomial

Recall:

$$\lambda \text{ is an eigenvalue of } A \Leftrightarrow A - \lambda I \text{ is not invertible}$$

So to find eigenvalues of A we solve

$$\det(A - \lambda I) = 0$$

The left hand side is a polynomial called the characteristic polynomial of A.

The roots of the characteristic polynomial are the eigenvalues of A.

Characteristic polynomial

The characteristic polynomial of $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ is:

$$\det\left(\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det \begin{pmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

$$= (5-\lambda)(1-\lambda) - 4$$

$$= \lambda^2 - 6\lambda + 1$$

$$= (\lambda - (3+2\sqrt{2}))(\lambda - (3-2\sqrt{2}))$$

So the eigenvalues are:

$$\lambda = \frac{6 \pm \sqrt{36-4}}{2}$$

$$= \frac{6 \pm \sqrt{32}}{2} = \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$$

Characteristic polynomial
2×2 matrices

The characteristic polynomial of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is:

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} -\lambda & 6 & 8 \\ 1/2 & -\lambda & 0 \\ 0 & 1/2 & -\lambda \end{pmatrix} = -\lambda(\lambda^2) - \frac{1}{2}\det\begin{pmatrix} 6 & 8 \\ 1/2 & -\lambda \end{pmatrix}$$

$$= -\lambda^3 - \frac{1}{2}(-6\lambda - 4)$$

$$= -\lambda^3 + 3\lambda + 2$$

$$= (\lambda-2)(\lambda+1)(\underset{-1}{\underline{a}}\lambda + \underset{-1}{\underline{b}})$$

$$= -(\lambda-2)(\lambda+1)^2$$

What are the eigenvalues?

2 (from before)

−1 (Ben)

Hint: We already know one eigenvalue!

Algebraic multiplicity

The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example. Find the algebraic multiplicities of the eigenvalues for

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\lambda^2(\lambda+1)(\lambda-1)$

$\lambda = 0$ has alg mult 2

$\lambda = 1, -1$ alg mult 1.

Similar matrices

Two $n \times n$ matrices A and B are similar if there is a matrix C so that

$$A = CBC^{-1}$$

Idea: A and B are doing the same thing, but with respect to different bases.

A, B Similar: $A = CBC^{-1}$

Fact. If A and B similar, they have the same characteristic polynomial.

and hence same eigenvalues.

Why?

char poly of A $= \det(A - \lambda I) = \det(CBC^{-1} - \lambda I)$

$= \det(CBC^{-1} - \lambda C I C^{-1})$

$= \det(C\underline{(B - \lambda I)}C^{-1})$

$= \cancel{\det(C)} \det(B - \lambda I) \cancel{\det(C^{-1})}$

Similar matrices

Example

Similar: $A = CBC^{-1}$

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

Idea: A and B are doing the same thing, but with respect to different bases.

Similar matrices

Example

Do a similar analysis of

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}$$

Eigenvectors and difference equations

Say we want to solve $x_{k+1} = Ax_k$. In other words, we need a sequence $x_0, x_1, x_2, \ldots$ with $x_1 = Ax_0$, $x_2 = Ax_1$, etc.

Example. $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a+b \end{pmatrix}.$

Structural engineering

Column buckling

Say we have a column with a compressive force. How will the column buckle?

Approximate column with finite number of points, say

$$(0,0),(0,1),(0,2),\ \ldots\ (0,5),(0,6)$$

Buckling leads to (roughly)

$$(0,0),(x_1,1),(x_2,2),\ \ldots\ (x_5,5),(0,6)$$

Engineers use a difference equation to model this

$$x_{i-1}-2x_i+x_{i+1}+\lambda x_i=0$$

Structural engineering
Column buckling

Difference equation for column buckling:

$$x_{i-1} - 2x_i + x_{i+1} + \lambda x_i = 0$$

We solve this difference equation as above. We need the eigenvector of

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

For most λ, the only eigenvector is 0, which corresponds to no buckling. This matrix has three eigenvalues, 1, 2, and 3.