Announcements Mar 2

- WebWork 2.8 and 2.9 due Thursday
- Homework 6 due Friday at the start of class
- Quiz 6 on 2.8 and 2.9 in class Friday
- Midterm 2 in class Friday Mar 11 on Chapters 2 & 3
- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
    - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
    - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
    - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Chapter 3

Determinants

Section 3.1

Introduction to Determinants

Where are we?

Remember:

> Almost every engineering problem, no matter how huge, can be reduced to linear algebra:
>
> $$Ax = b \quad \text{or} \quad \text{linear system}$$
> $$Ax = \lambda x \quad \text{eigenvalue problem}$$

We have said most of what we are going to say about the first problem. We are now aiming towards the second problem.

Outline

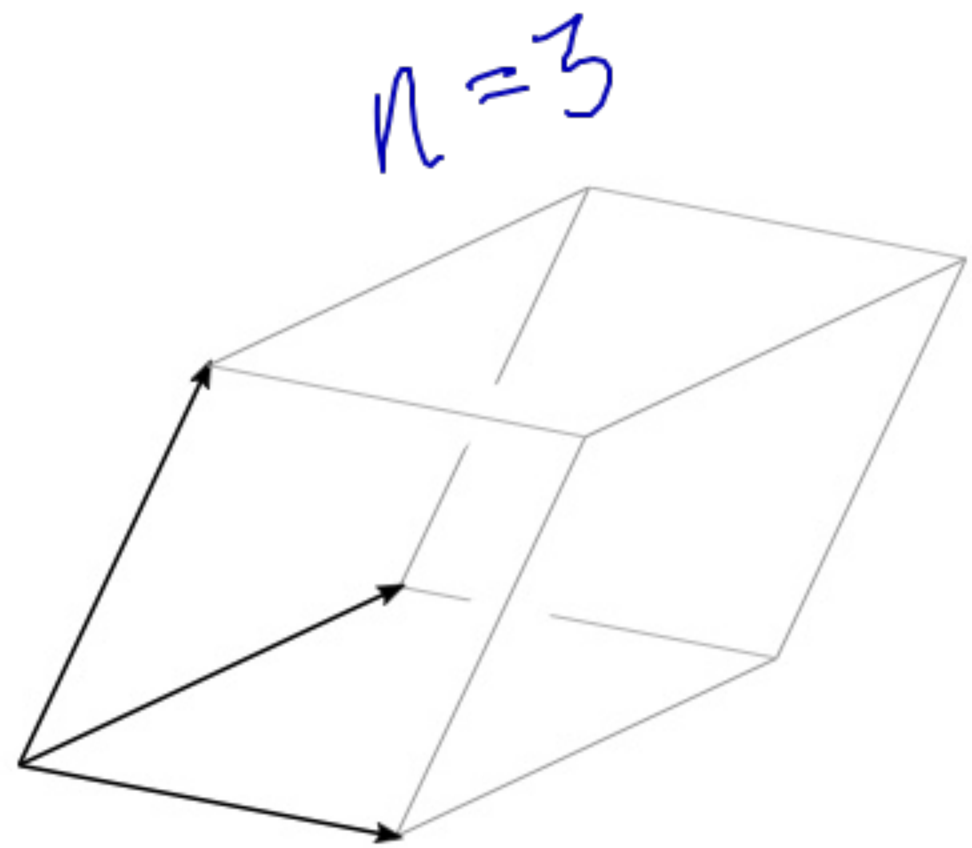
- The idea of the determinant
- A formula for the determinant
- More formulas for the determinant
- Determinants of triangular matrices
- A formula for the inverse of a matrix

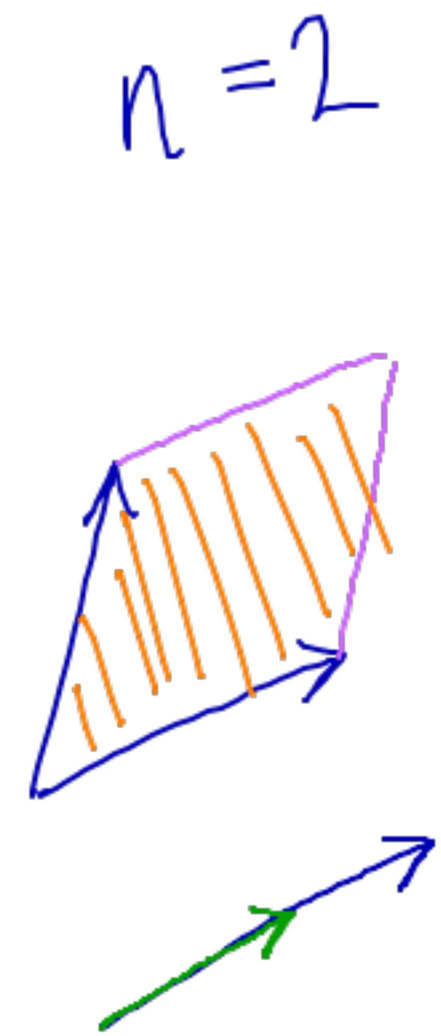
The idea of determinant

Let A be an $n \times n$ matrix.

$\rightsquigarrow$ n vectors in $\mathbb{R}^n$

$\rightsquigarrow$ a parallelepiped P:





$\rightsquigarrow$ volume

Idea: A is invertible $\Leftrightarrow$ the volume of P is... not 0.

The idea of determinant

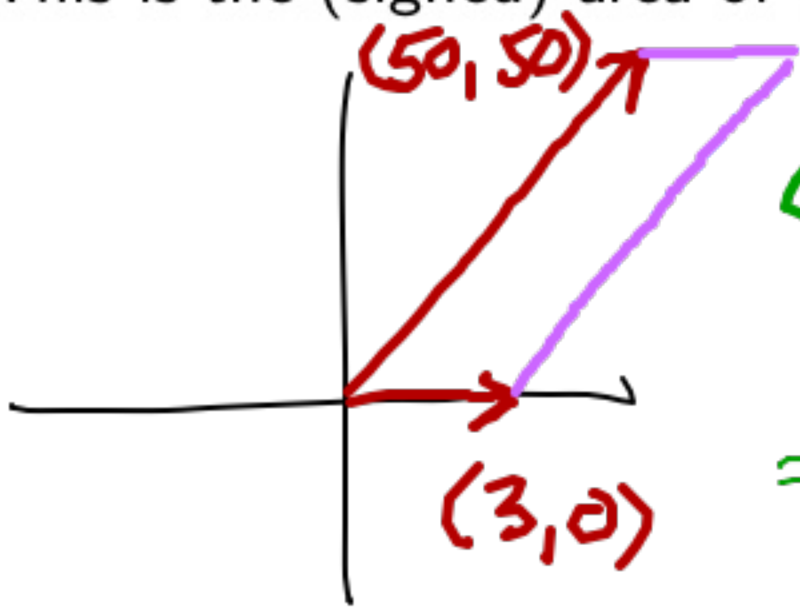
Idea: A is invertible $\Leftrightarrow$ the volume of P is nonzero

The determinant is a number $\det(A)$ whose absolute value is the volume of P.

For 2×2 matrices we already have a formula:

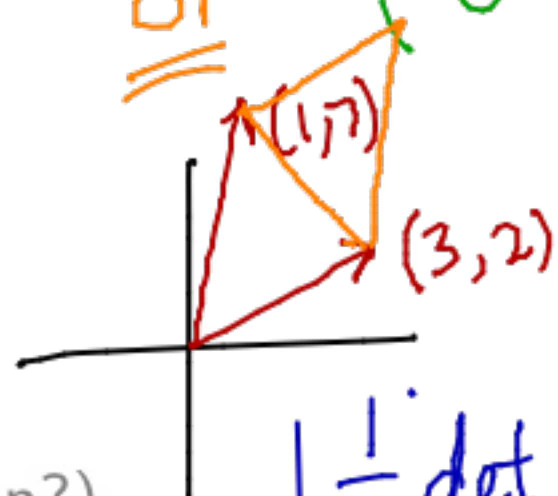
$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

This is the (signed) area of the parallelogram spanned by the columns. Try it!



area = base · height $= 3 \cdot 50 = 150.$

or $\det\begin{pmatrix} 3 & 50 \\ 0 & 50 \end{pmatrix} = 150.$



Area of triangle!

$\left|\frac{1}{2}\det\begin{pmatrix} 1 & 3 \\ 7 & 2 \end{pmatrix}\right| = \left|\frac{1}{2}(-19)\right|$

(What does the sign of the determinant mean?)

The idea of determinant

Let's do a reality check. We wanted:

$$A \text{ is invertible} \Leftrightarrow \det(A) \neq 0$$

Let's row reduce:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightsquigarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{11}a_{21} & a_{11}a_{22} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{a_{11}} & a_{12} \\ 0 & \boxed{a_{11}a_{22} - a_{21}a_{12}} \end{pmatrix}$$

A formula for the determinant

We will give a recursive formula.

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $A_{12} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$ $A_{31} = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$

First some terminology:

$$\begin{aligned} A_{ij} &= ij\text{th minor of } A \\ &= (n-1)\times(n-1) \text{ matrix obtained by deleting the } i\text{th row and } j\text{th column} \end{aligned}$$

$C_{12} = (-1)^{1+2} \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} = -(-6) = 6$

$$\begin{aligned} C_{ij} &= (-1)^{i+j} \det(A_{ij}) \\ &= ij\text{th cofactor of } A \end{aligned}$$

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$C_{31} = (-1)^{3+1} \det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = +(-3) = -3$

Finally:

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}$$

$$= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

A formula for the determinant

The recursive formula:

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}$$

Need to start somewhere...

1×1 matrices

$$\det(a_{11}) = a_{11}$$

2×2 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} C_{11} + a_{12} C_{12}$$

$$= a_{11} (-1)^{1+1} \det A_{11} + a_{12} (-1)^{1+2} \det A_{12}$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

A formula for the determinant

3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= +a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$- a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

$$+ a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Determinants

Compute

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = 5 \cdot \det\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$$

$$- 1 \cdot \det\begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix}$$

$$+ 0$$

$$= 5(-3) - 1(-7)$$

$$= -8.$$

A formula for the determinant

Another formula for 3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Use this formula to compute

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = -15 + 8 + 0$$
$$-0 - 1 - 0 = -8$$

Expanding across other rows and columns

The formula we gave for $\det(A)$ is the expansion across the first row. It turns out you can compute the determinant by expanding across any row or column:

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} \text{ for any fixed } i$$

$$= \sum_{i=1}^{n} a_{ij}C_{ij} \text{ for any fixed } j$$

Compute:

$$\det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix} = +0 - 0 + 1 \cdot \det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 1$$

$$= -1 \cdot \det\begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix} + 1 \cdot \det\begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad = -1 \cdot 1 + 1 \cdot 2 = 1$$

Determinants of triangular matrices

If A is upper (or lower) triangular, $\det(A)$ is easy to compute:

$$\det\begin{pmatrix} 2 & 1 & 5 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix} = 2\det\begin{pmatrix} 1 & 2 & -3 \\ 0 & 5 & 9 \\ 0 & 0 & 10 \end{pmatrix}$$

$$= 2 \cdot 1 \cdot \det\begin{pmatrix} 5 & 9 \\ 0 & 10 \end{pmatrix}$$

$$= 2 \cdot 1 \cdot 5 \cdot 10 = 100.$$

$\|$

product of diag entries

A formula for the inverse

(from Section 3.3)

2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad A^{-1} = \frac{1}{\cancel{ad-bc}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

det(A)

$n \times n$ matrices

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{pmatrix}$$

$$= \frac{1}{\det(A)} (C_{ij})^T$$

Check that these agree!

cofactor matrix: $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \xrightarrow{\text{transpose}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\rightarrow A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The proof uses Cramer's rule (see the notes on the course home page).

A formula for the inverse

(from Section 3.3)

$n \times n$ matrices

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \ldots & C_{nn} \end{pmatrix}$$
$$= \frac{1}{\det(A)} (C_{ij})^T$$

Compute:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$$