Announcements Mar 28

- WebWork 5.2 and 5.3 due Thursday
- Quiz 8 on 5.2 and 5.3 on Friday
- Homework 7 due Friday April 8
- Midterm 3 in class Friday April 8 on Chapter 5

- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236

- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4

- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6
Section 5.3
Diagonalization
5.3 Diagonalization

Outline

- Taking powers of diagonal matrices is easy
- Taking powers of diagonalizable matrices is still easy
- Algebraic multiplicity vs geometric multiplicity vs diagonalizability
- Application: networks

\[(x-1)^2\]
Powers of diagonal matrices

We have seen that it is useful to take powers of matrices: for instance in computing rabbit populations.

\[ \mathbf{v}, \mathbf{A}\mathbf{v}, \mathbf{A}^2\mathbf{v}, \mathbf{A}^3\mathbf{v} \]

If \( \mathbf{A} \) is diagonal, \( \mathbf{A}^k \) is easy to compute. For example:

\[
\begin{pmatrix}
2 & 0 \\
0 & 3
\end{pmatrix}^{10} = \begin{pmatrix}
2^{10} & 0 \\
0 & 3^{10}
\end{pmatrix}
\]
Powers of matrices that are similar to diagonal ones

What if $A$ is not diagonal? Suppose we need to compute

$$
\begin{pmatrix}
1 & 2 \\
-1 & 4
\end{pmatrix}^{10}
$$

What would we do?

Earlier in the notes, we saw this matrix is similar to a diagonal one:

$$
\begin{pmatrix}
1 & 2 \\
-1 & 4
\end{pmatrix} = \begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
2 & 0 \\
0 & 3
\end{pmatrix} \begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}^{-1}
$$

“diagonalization”

So...

$$
A^2 = (C B C^{-1})(C B C^{-1}) = C B^2 C^{-1}
$$

$$
A^{10} = C B^{10} C^{-1} = \begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
2^{10} & 0 \\
0 & 3^{10}
\end{pmatrix} \begin{pmatrix}
1 & -1 \\
-1 & 2
\end{pmatrix}
$$

$$
A^3 = C B C^{-1} B C^{-1} C B C^{-1} = C B^3 C^{-1}
$$
Diagonalization

Suppose $A$ is $n \times n$. We say that $A$ is diagonalizable if it is similar to a diagonal matrix:

$$A = CDC^{-1}$$  \hspace{2cm} D = \text{diagonal}$$

How does this factorization of $A$ help describe what $A$ does to $\mathbb{R}^n$?

$$
\begin{pmatrix}
1 & 2 \\
-1 & 4
\end{pmatrix} = 
\begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}^{-1}
$$

$C(e_1) = (2)$
$C(e_2) = (1)$
$C^{-1}(2) = e_1$
$C^{-1}(1) = e_2$

eigenspaces for $A$
eigenvectors
eigenvalues

eigenspaces for $A$
**Theorem.** A is diagonalizable \( \iff \) \( A \) has \( n \) linearly independent eigenvectors.

In this case

\[
A = (v_1 \ v_2 \ \cdots \ v_n) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} (v_1 \ v_2 \ \cdots \ v_n)^{-1}
\]

where \( v_1, \ldots, v_n \) are linearly independent eigenvectors and \( \lambda_1, \ldots, \lambda_n \) are the corresponding eigenvalues (in *order*).

Why?
Diagonalization

Fact. If $A$ is diagonalizable, bases for the eigenspaces give a basis for $\mathbb{R}^n$.

Why?
Example

Diagonalize if possible.

\[
\begin{pmatrix}
2 & 6 \\
0 & -1
\end{pmatrix}
\]

**Eigenvalues**

\[
\det \begin{pmatrix}
2 - \lambda & 6 \\
0 & -1 - \lambda
\end{pmatrix} \rightarrow \lambda = 2, -1
\]

**Distinct.**

**Eigenvalues**

\[
\begin{pmatrix}
\lambda = 2 \\
0 & 6 \\
0 & -3
\end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\
0 & 0
\end{pmatrix} \rightarrow \text{eigenvector} \begin{pmatrix} 1 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
\lambda = -1 \\
3 & 6 \\
0 & 0
\end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\
0 & 0
\end{pmatrix} \rightarrow \text{eigenv.} \begin{pmatrix} -2 \\
1
\end{pmatrix}
\]

\[
x_1 + 2x_2 = 0 \\
x_2 \text{ free}, x_1 = -2x_2
\]

**Finally:**

\[
\begin{pmatrix}
2 & 6 \\
0 & -1
\end{pmatrix} = \begin{pmatrix} 1 & -2 \\
0 & 1
\end{pmatrix} \begin{pmatrix} 2 & 0 \\
0 & -1
\end{pmatrix} \begin{pmatrix} 1 & -2 \\
0 & 1
\end{pmatrix}^{-1}
\]
Example

Diagonalize if possible.

\[
\begin{pmatrix}
3 & 1 \\
0 & 3 \\
\end{pmatrix}
\]

eigenvalues: \(3\) (alg mult. = 2)

eigenvectors: 
\[
\begin{pmatrix}
0 & 1 \\
0 & 0 \\
\end{pmatrix} \rightarrow 
\begin{pmatrix}
1 \\
0 \\
\end{pmatrix}
\]

Another example

\[
A = \begin{pmatrix}
3 & 0 \\
0 & 3 \\
\end{pmatrix}
\]

\(\lambda = 3\) (alg. mult. 2)

eigenvectors: 
\[
(1, 1)^T
\]

only eigenvector
up to scale

\[\rightarrow\] not diagonalizable.
More Examples

Diagonalize if possible.

\[
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
a & b \\
b & a
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
2 & 0 & 0 \\
1 & 2 & 1 \\
-1 & 0 & 1
\end{pmatrix}
\]
Poll

Which are true?
1. if $A$ is diagonalizable then $A^2$ is
2. if $A$ is diagonalizable then $A^{-1}$ is
3. if $A^2$ is diagonalizable then $A$ is
4. if $A$ is diagonalizable and $B$ is similar to $A$ then $B$ is

1. $A = CDC^{-1} \implies A^2 = CD^2C^{-1}$
2. $A^{-1} = CD^{-1}C^{-1}$ as long as $0$ not an eigenval.
3. $A^2 = CDC^{-1} \implies A = "CD^{1/2}C^{-1}"

4. $A = CDC^{-1}$ $B = PAP^{-1}$
   $\implies B = PCDC^{-1}P^{-1}$
   $= QDQ^{-1}$ $Q = PC$
Distinct Eigenvalues

Fact. If $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.

Why?

$n$ distinct eigenvalues

$\Rightarrow$ $n$ lin ind eigenvectors.
Non-Distinct Eigenvalues

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- $a_i$ = algebraic multiplicity of $\lambda_i$
- $d_i$ = dimension of $\lambda_i$ eigenspace ("geometric multiplicity")

Then

1. $d_i \leq a_i$ for all $i$ \textcolor{red}{actually, } 1 \leq d_i \leq a_i$
2. $A$ is diagonalizable $\iff \Sigma d_i = n$
   ($\iff \Sigma a_i = n$ and $d_i = a_i$ for all $i$)
Example

Use your diagonalization of \[
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}
\] to find a formula for the \(n^{th}\) Fibonacci number.
Application: Social Networks

Consider the social network below.

- We want to find communities, say, a group of people so there is a direct path connecting any two.
- Make a matrix, $M$, whose $ij$-entry is the number of arrows from $i$ to $j$.
- Then the $ij$ entry of $M^2$ is the number of paths of length 2 to $i$ to $j$.
  Why?
- Similar for $M^3$, etc.
- So the $ij$ entry of $M + M^2 + \cdots M^k$ is the number of paths of length at most $k$. We look for positive minors.

The leading eigenvalue is a measure of how connected the network is.
Application: Business

Say your car rental company has 3 locations. Make a matrix $M$ whose $ij$ entry is the probability that a car at location $i$ ends at location $j$. For example,

$$M = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

Note the columns sum to 1. The eigenvector with eigenvalue 1 is the steady-state. Any other vector gets pulled to this state. Applying powers of $M$ gives the state after some number of iterations.