Announcements Mar 28

- WebWork 5.2 and 5.3 due Thursday
- Quiz 8 on 5.2 and 5.3 on Friday
- Homework 7 due Friday April 8 HW 8 due April 1.
- Midterm 3 in class Friday April 8 on Chapter 5
- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 5.3

Diagonalization

5.3 Diagonalization

Outline

- Taking powers of diagonal matrices is easy
- Taking powers of diagonalizable matrices is still easy
- Algebraic multiplicity vs geometric multiplicity vs diagonalizability
- Application: networks

$(x-1)^2$

Powers of diagonal matrices

We have see that it is useful to take powers of matrices: for instance in computing rabbit populations.

v, Av, A^2v, A^3v

If A is diagonal, A^k is easy to compute. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{10} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{pmatrix}$$

Powers of matrices that are similar to diagonal ones

What if A is not diagonal? Suppose we need to compute

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}^{10}$$

What would we do?

A similar to B: $A = CBC^{-1}$

Earlier in the notes, we saw this matrix is similar to a diagonal one:

$$\boxed{\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \boxed{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$ "diagonalization"

$A \quad = \quad C \quad B \quad C^{-1}$

So...

$$A^{2} = (CBC^{-1})(CBC^{-1}) = CB^{2}C^{-1}$$

$$A^{10} = CB^{10}C^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\left[A^3 = CB\cancel{C}^{-1}\cancel{C}B\cancel{C}^{-1}\cancel{C}BC^{-1} = CB^3C^{-1} \right]$$

Diagonalization

Suppose A is $n \times n$. We say that A is diagonalizable if it is similar to a diagonal matrix:

$$A = CDC^{-1} \qquad\qquad D = \text{diagonal}$$

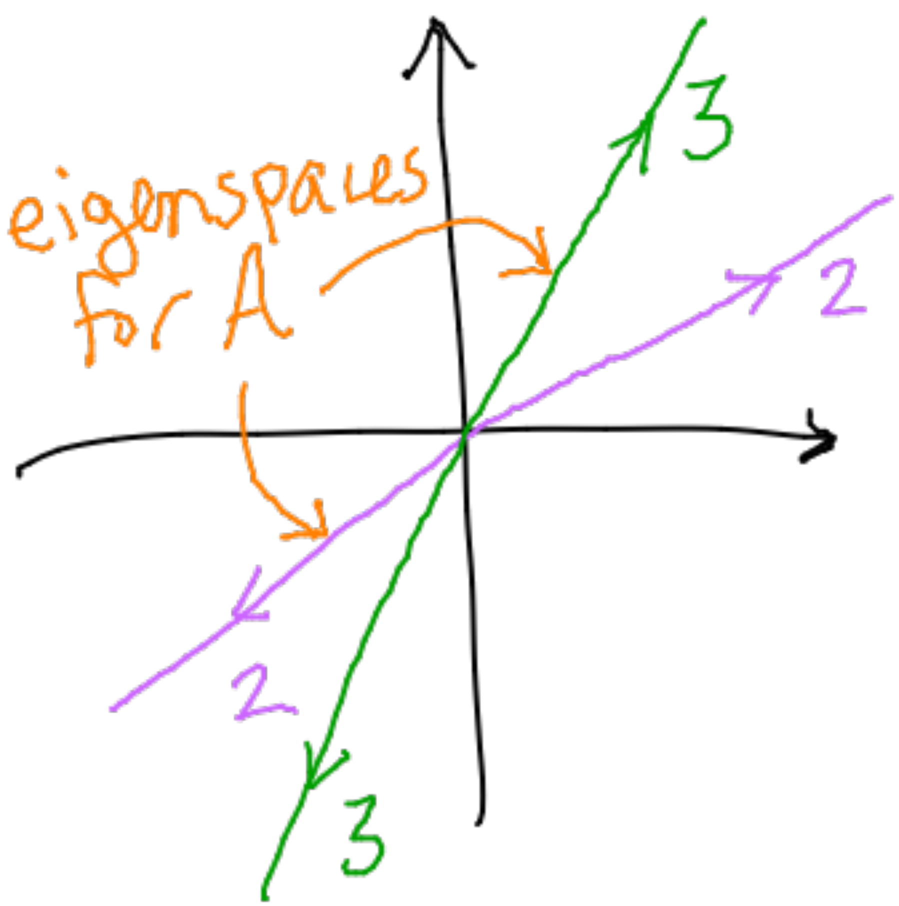
How does this factorization of A help describe what A does to $\mathbb{R}^n$?

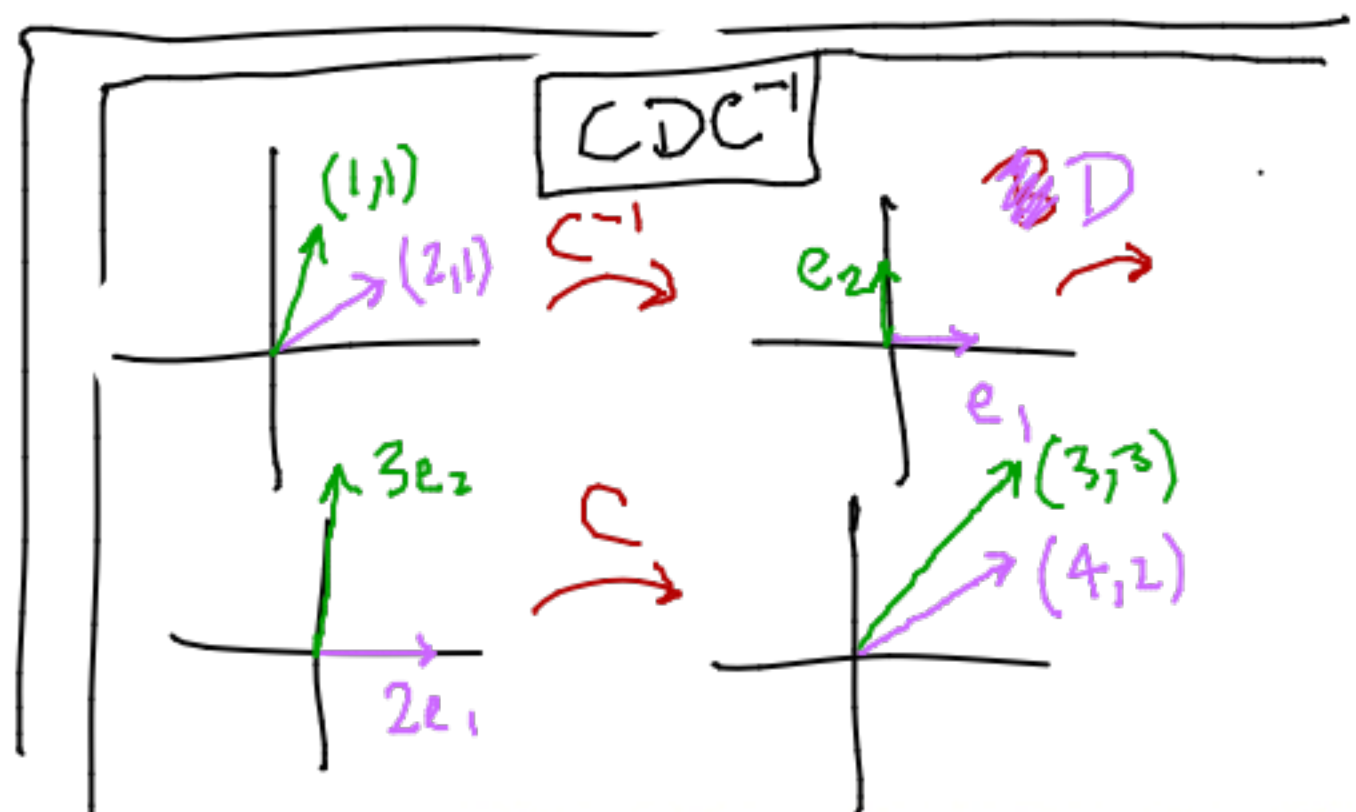
$$A = \underbrace{\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}}_{} = \overset{C}{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}} \overset{D}{\underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}}_{\text{eigenvals}}} \overset{C^{-1}}{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}}$$

eigenvectors

$$C(e_1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad C(e_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$C^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = e_1 \qquad C^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = e_2$$





Diagonalization

Theorem. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

In this case

$$C \qquad D \qquad C^{-1}$$

$$A = (v_1 \; v_2 \cdots v_n) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} (v_1 \; v_2 \cdots v_n)^{-1}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues (in order).

Why?

Diagonalization

Fact. If A is diagonalizable, bases for the eigenspaces give a basis for $\mathbb{R}^n$.

Why?

Example

Diagonalize if possible.

$$\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$$

eigenvector   eigenvector

Eigenvalues $\det\begin{pmatrix} 2-\lambda & 6 \\ 0 & -1-\lambda \end{pmatrix} \rightsquigarrow \lambda = 2, -1$ distinct.

Eigenvectors $\boxed{\lambda = 2}$ $\begin{pmatrix} 0 & 6 \\ 0 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & \boxed{1} \\ 0 & 0 \end{pmatrix} \rightsquigarrow$ eigen vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\boxed{\lambda = -1}$ $\begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \rightsquigarrow$ eigenv. $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$x_1 + 2x_2 = 0$

x_2 free, $x_1 = -2x_2$

Finally: $$\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{-1}$$

Example

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

eigenvalues: 3 (alg mult. = 2)

eigenvectors: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

<u>only</u> eigenvector up to scale

$\rightsquigarrow$ not diagonalizable.

Another example

$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$\lambda = 3$ (alg. mult 2) eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

More Examples

Diagonalize if possible.

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Poll

$$\begin{pmatrix}2&0\\0&3\end{pmatrix}^{-1} = \begin{pmatrix}1/2&0\\0&1/3\end{pmatrix}$$

Poll

Which are true?

1. if A is diagonalizable then A^2 is
2. if A is diagonalizable then A^{-1} is
3. if A^2 is diagonalizable then A is
4. if A is diagonalizable and B is similar to A then B is

1. $A = CDC^{-1} \rightsquigarrow A^2 = CD^2C^{-1}$
2. $A^{-1} = CD^{-1}C^{-1}$ as long as 0 not an eigenval.
3. $A^2 = CDC^{-1} \rightsquigarrow \boxed{A = \text{"}CD^{1/2}C^{-1}\text{"}}$
4. $A = CDC^{-1}$ $\quad B = PAP^{-1}$

$$\rightsquigarrow B = PCDC^{-1}P^{-1} = QDQ^{-1} \qquad Q = PC$$

Distinct Eigenvalues

Fact. If A has n distinct eigenvalues, then A is diagonalizable.

Why?

n distinct eigenvalues

$\rightsquigarrow$ n lin ind eigenvectors.

Non-Distinct Eigenvalues

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity")

Then

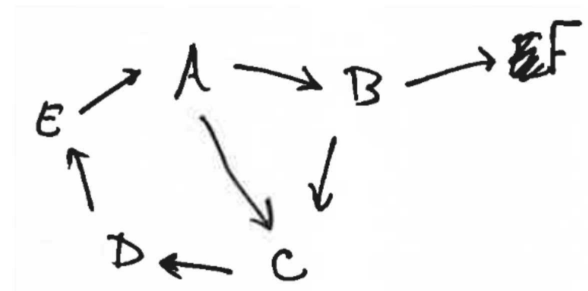
1. $d_i \leq a_i$ for all i actually. $1 \leq d_i \leq a_i$
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n$
   ($\Leftrightarrow \Sigma a_i = n$ and $d_i = a_i$ for all i)

Example

Use your diagonalization of $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ to find a formula for the n^{th} Fibonacci number.

Application: Social Networks

Consider the social network below.



- We want to find communities, say, a group of people so there is a direct path connecting any two.
- Make a matrix, M, whose ij-entry is the number of arrows from i to j.
- Then the ij entry of M^2 is the number of paths of length 2 to i to j. Why?
- Similar for M^3, etc.
- So the ij entry of $M + M^2 + \cdots M^k$ is the number of paths of length at most k. We look for positive minors.

The leading eigenvalue is a measure of how connected the network is.

Application: Business

Say your car rental company has 3 locations. Make a matrix M whose ij entry is the probability that a car at location i ends at location j. For example,

$$M = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

Note the columns sum to 1. The eigenvector with eigenvalue 1 is the steady-state. Any other vector gets pulled to this state. Applying powers of M gives the state after some number of iterations.