Announcements Mar 30

- WebWork 5.2 and 5.3 due Thursday
- Homework 8 due Friday in class
- Quiz 8 on 5.2 and 5.3 on Friday
- Homework 7 due Friday April 8
- Midterm 3 in class Friday April 8 on Chapter 5
- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 5.5

Complex Eigenvalues

Outline

- Rotation matrices have no eigenvectors
- Crash course in complex numbers
- Finding complex eigenvectors and eigenvalues
- Complex eigenvalues correspond to rotations + dilations

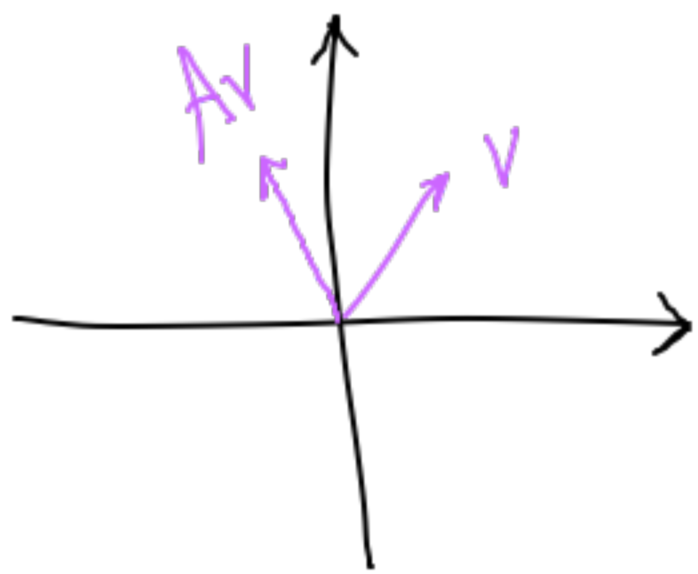
A matrix without an eigenvector

Recall the rotation matrix:

$$A = 1/\sqrt{2}\boxed{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}$$

rotation by $\pi/4$

This matrix has no eigenvectors. Why?



or $\det\begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1$

$= \lambda^2 - 2\lambda + 2$

$\rightsquigarrow \lambda = \dfrac{2 \pm \sqrt{-4}}{2} = \dfrac{2 \pm \sqrt{4}\sqrt{-1}}{2} = \dfrac{2 \pm 2i}{2} = 1 \pm i$

Imaginary numbers

Problem. When solving polynomial equations, we often run up against the issue that we can't take the square root of a negative number:

$$x^2 + 1 = 0 \rightsquigarrow x^2 = -1 \rightsquigarrow x = \pm\sqrt{-1}$$

Solution. Take square roots of negative numbers:

$$x = \pm\sqrt{-1}$$

$$x = \sqrt{-1} = i$$

We usually write $\sqrt{-1}$ as i (for "imaginary"), so $x = \pm i$.

$$x^2 = -1$$

$$x = -\sqrt{-1}$$

$$x^2 = (-\sqrt{-1})^2$$

$$= -1$$

Now try solving these:

$$\boxed{i^2 = -1}$$

$$x^2 + 3 = 0$$

$$x^2 = -3$$

$$x = \pm\sqrt{-3} = \pm\sqrt{3}\sqrt{-1}$$

$$= \pm\sqrt{3}\, i$$

$$x^2 - x + 1 = 0$$

$$x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}\, i}{2}$$

Complex numbers

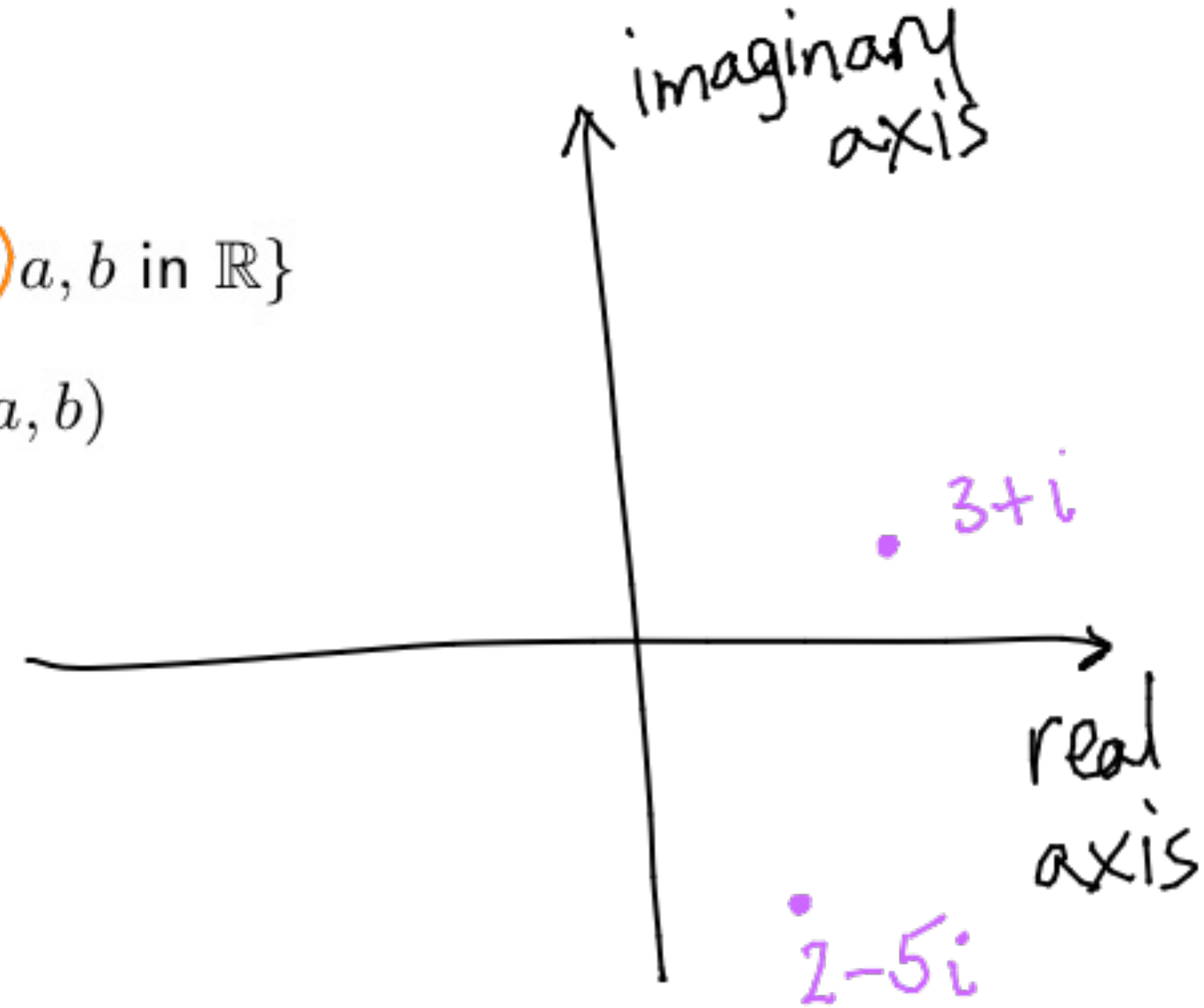
The complex numbers are the numbers

$3+i$

$2-5i$

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can identify $\mathbb{C}$ with $\mathbb{R}^2$: $\quad a + bi \leftrightarrow (a, b)$



We can add/multiply complex numbers:

$(2 - 3i) + (-1 + i) = 1 - 2i$ like: $(2,-3)+(-1,1)=(1,-2)$

$$(2 - 3i)(-1 + i) = -2 + 3i + 2i - 3i^2$$
$$= -2 + 5i + 3$$
$$= 1 + 5i$$

Complex numbers

The complex numbers are the numbers

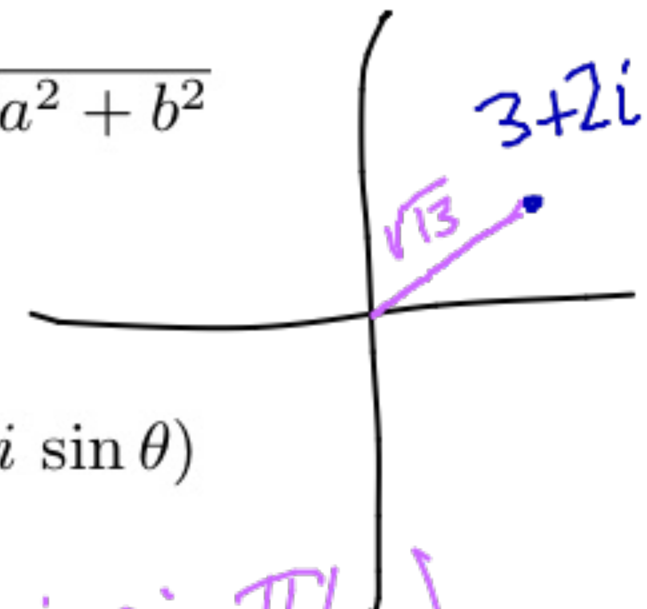
$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can conjugate complex numbers: $\overline{a+bi} = a - bi$

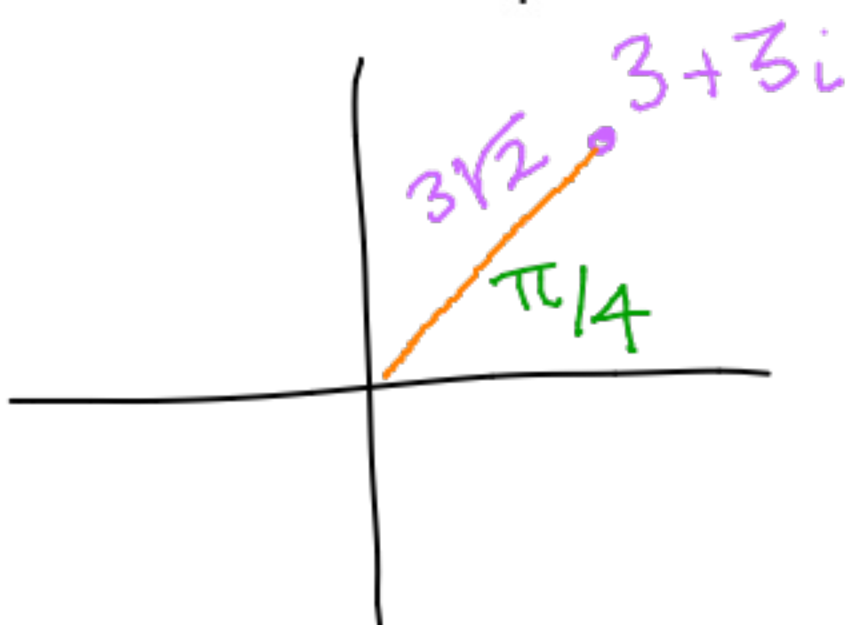
$$\overline{3+7i} = 3-7i$$

We can take absolute values of complex numbers: $|a+bi| = \sqrt{a^2+b^2}$

$$|3+2i| = \sqrt{9+4} = \sqrt{13}$$



We can write complex numbers in polar coordinates: $r(\cos\theta + i\,\sin\theta)$



$$3\sqrt{2}\left(\cos \pi/4 + i \sin \pi/4\right)$$

Complex numbers and polynomials

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

Fact. Every quadratic polynomial has two complex roots. (or real)

Fundamental theorem of algebra. Every polynomial of degree n has exactly n complex roots.

We can now find complex eigenvectors and eigenvalues.

Fact. If λ is an eigenvalue of A with eigenvector v then $\bar{\lambda}$ is an eigenvalue of A with eigenvector $\bar{v}$.

Why?

$$Av = \lambda v$$

$$A\bar{v} = \overline{Av} = \overline{\lambda v} = \bar{\lambda}\bar{v}$$

A matrix with an eigenvector

Find the eigenvectors and eigenvalues of:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is rotation by $\pi/4$

From before: $\lambda = 1 \pm i$

$\underline{\underline{\lambda = 1+i}}$ $\begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{-i} & \boxed{-1} \\ 0 & 0 \end{pmatrix}$

$\rightsquigarrow$ eigenvector: $\begin{pmatrix} \boxed{-1} \\ \boxed{i} \end{pmatrix} = v$

$\begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix}$

$\rightsquigarrow$ evector $\begin{pmatrix} -7 \\ 3 \end{pmatrix}$

or $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$

$\underline{\underline{\lambda = 1-i}}$ $\bar{v} = \begin{pmatrix} \overline{-1+0i} \\ \overline{0+i} \end{pmatrix} = \begin{pmatrix} -1 \\ -i \end{pmatrix}$

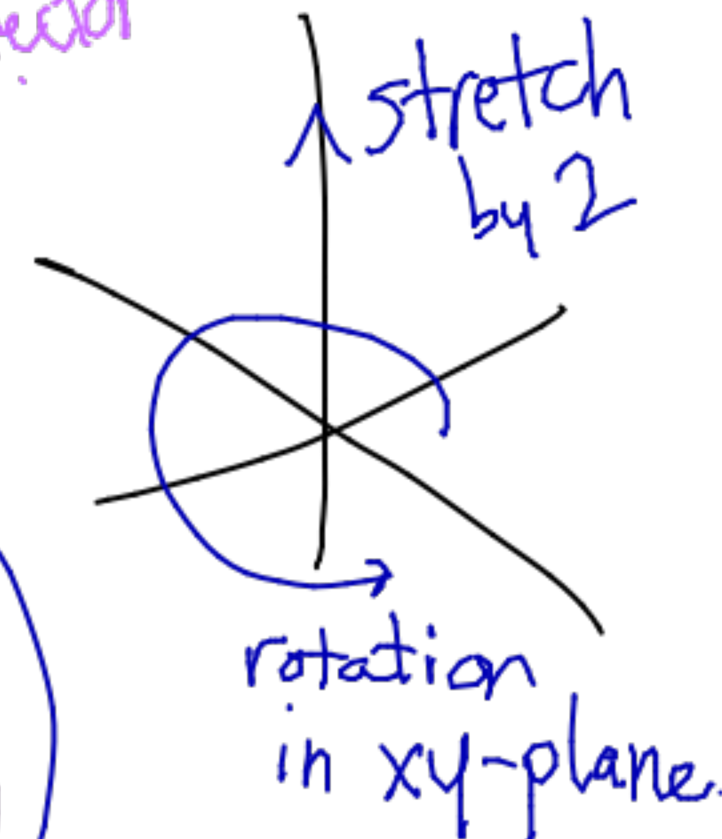
A 3×3 example

Find the eigenvectors and eigenvalues of:

2- eigenvector

e_3

$$A = \begin{pmatrix} 4/5 & -3/5 & 0 \\ 3/5 & 4/5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



$$\det \begin{pmatrix} 4/5-\lambda & -3/5 & 0 \\ 3/5 & 4/5-\lambda & 0 \\ \boxed{0} & \boxed{0} & \boxed{2-\lambda} \end{pmatrix}$$

$$= (2-\lambda)\left((4/5-\lambda)(4/5-\lambda) + 9/25 \right)$$

$$= (2-\lambda)\left(\frac{16}{25} - \frac{8}{5}\lambda + \lambda^2 + 9/25 \right)$$

$$= (2-\lambda)\left(\lambda^2 - \frac{8}{5}\lambda + 1 \right)$$

$\lambda = 2$

$$\lambda = \frac{\frac{8}{5} \pm \sqrt{\frac{64}{25} - \frac{100}{25}}}{2} = \frac{4}{5} \pm \frac{3}{5} i$$

$$\lambda = \frac{4}{5} + \frac{3}{5} i$$

$$\begin{pmatrix} -3/5\, i & -3/5 & 0 \\ 3/5 & -3/5\, i & 0 \\ 0 & 0 & \frac{6}{5} - \frac{3}{5} i \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} i & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \cancel{2-i}\ 1 \end{pmatrix}$$

$$\rightsquigarrow \text{evector} = \begin{pmatrix} -1 \\ i \\ 0 \end{pmatrix}$$

What do complex eigenvalues mean?

With n real eigenvectors, we have a picture for what the matrix does to $\mathbb{R}^n$.

What about complex eigenvectors? What does the matrix do to $\mathbb{R}^n$?

We saw that rotation matrices have complex eigenvalues. Do complex eigenvalues always correspond to rotations?

Almost...

Fact. If an $n \times n$ matrix A has a complex eigenvalue there is a 2D plane in $\mathbb{R}^n$ where A is (similar to) the product of a rotation and a dilation.

What do complex eigenvalues mean?

Fact. If an $n \times n$ matrix A has a complex eigenvalue there is a 2D plane in $\mathbb{R}^n$ where A is (similar to) the product of a rotation and a dilation.

Here is the actual statement for 2×2 matrices:

Theorem. Let A be a matrix with a complex eigenvalue $\lambda = a + bi$ (where $b \neq 0$) and associated eigenvector v. Then

$$A = PCP^{-1}$$

where

$$P = (\mathrm{Re}\, v \quad \mathrm{Im}\, v) \quad \text{and} \quad C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

If we write $a + bi$ as $r(\cos\theta + i\,\sin\theta)$ then C is the composition of a rotation by θ and scaling by r.