Section 3.2

Properties of Determinants

Where are we?

Last time, we also gave a formula for the determinant and said that the absolute value of the determinant of A is the volume of the parallelepiped spanned by the columns of A.

So: A is invertible $\Leftrightarrow \det(A) \neq 0$

Remaining questions:

- Where do the formulas for determinant come from?
- Why do the formulas tell us about volume?
- How can we compute determinants more efficiently?

Outline

- A definition of determinant in terms of row operations
- Using the definition of determinant to compute the determinant
- Determinants of products: $\det(AB)$
- Determinants and linear transformations

A definition of determinant in terms of row operations

A determinant is a function

$$\det : \{\text{matrices}\} \to \mathbb{R}$$

with the following properties:

1. $\det(I_n) = 1$
2. If we do a row replacement on a matrix, the determinant is unchanged
3. If we swap two rows of a matrix, the determinant scales by -1
4. If we scale a row of a matrix by k, the determinant scales by k

Why would we think of this? *Answer: This is exactly how volume works.*

Try it out for 2×2 matrices:

A definition of determinant in terms of row operations

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Problem. Just using these rules, compute the determinants:

$$\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

Determinants and invertibility

Fact. If A has a zero row, then $\det(A) = 0$.

Why? *Hint: use a row scale by 0.*

Theorem. A is invertible $\Leftrightarrow \det(A) \neq 0$

Why?

Computing determinants

...using the definition in terms of row operations

$$\det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix} =$$

A Mathematical Conundrum

We have this nice definition of a determinant, and it gives us a fast way to compute determinants (much fast than cofactor expansion for a large matrix), but...

We don't know that such a determinant function exists?

More specifically, we haven't ruled out the possibility that two different row reductions might gives us two different answers for the determinant.

Don't worry! It will all be okay.

Properties of the determinant

Let $\det : \{\text{matrices}\} \to \mathbb{R}$ be a function with the above four properties.

Fact 1. There is such a function $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$.

Fact 3. If we row reduce A without row scaling then

$$\det(A) = (-1)^{\text{swaps}}(\text{product of diagonal entries of REF}).$$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

Fact 5. $\det(AB) = \det(A)\det(B)$

Fact 6. $|\det(A)|$ is the volume of the parallelepiped spanned by the cols of A.

If you want the proofs, see the course web site. Actually Fact 1 is the hardest!

Powers

Fact 5. $\det(AB) = \det(A)\det(B)$

Use this fact to compute

$$\det\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix}^5\right)$$

Powers

Poll

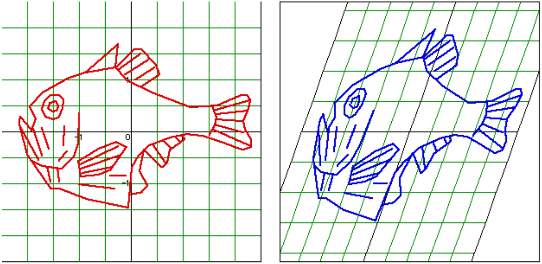
Suppose we know A^5 is invertible. Is A invertible?

1. yes
2. no
3. maybe

Determinants and linear transformations

Fact 7. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T_A(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

This works even if S is curvy, like a circle or an ellipse, or:



Why? First check that it works for little squares/cubes. Then: Calculus!