Chapter 2

Review
Chapter 2 Review
Matrix Multiplication and Inverses

Products
\[ AB \neq BA \]
\[ T_{AB} = T_A \circ T_B \]

Inverses
- if \( AB = BA = I \), then \( B = A^{-1} \)
- \( T_A^{-1} = T_{A^{-1}} \)
- \( Ax = b \Leftrightarrow x = A^{-1}b \)
- Find \( A^{-1} \) by row reducing \((A|I)\) to get \((I|A^{-1})\)
- Another perspective: \( E_k E_{k-1} \cdots E_2 E_1 A = I \), where each \( E_i \) is an elementary matrix. This implies that \( E_k E_{k-1} \cdots E_2 E_1 = A^{-1} \).
Chapter 2 Review
Invertible Matrix Theorem

Let $A$ be an $n \times n$ matrix. The following are equivalent:

- $A$ is invertible
- $A$ can be reduced to $I$
- $A$ has $n$ pivots
- $\text{Nul}(A) = \{0\}$
- $T_A$ is one-to-one
- $T_A$ is onto
- $Ax = b$ is consistent for all $b$
- $\dim \text{Col}(A) = n$

etc ...
Let $A = LU$, where $L$ is lower triangular, $U$ is in echelon form.

- Easy to solve for many $b$.
- Algorithm
  - Step 1. Solve $Ly = b$
  - Step 2. Solve $Ux = y$
- To find $L$ and $U$, row reduce $A$ to echelon form to obtain $U$, then use negative of row operations to obtain $L$. Be careful:
  - go column by column
  - only use lower row replacement
Chapter 2 Review

Subspaces

- A **subspace** is a non-empty subset of $\mathbb{R}^n$ closed under linear combinations.
- Two important subspaces are
  - $\text{Col}(A) = \text{span of columns of } A$.
  - $\text{Nul}(A) = \text{(solutions to } Ax = 0)$. 
- A **basis** for a subspace $W$ is a set of lin. ind. vectors that spans $W$.
  - To find the $B$–coords of $u$, solve $Bx = u$
- The **dimension** of a subspace is the number of elements in the basis.
- Use row reduction to find a basis for $\text{Col}(A)$ or $\text{Nul}(A)$.
  - Pivot columns are used to identify the basis for $\text{Col}(A)$.
  - Parametric form is used to identify the basis for $\text{Nul}(A)$.

**Note.** $\text{Col}(A) = \text{range of } T_A$
Chapter 2 Review

**Rank Theorem** Let $A$ be an $m \times n$ matrix, then

$$\dim \text{Col}(A) + \dim \text{Nul}(A) = n.$$ 

**Basis Theorem**

Suppose $V$ is a $k$-dimensional subspace of $\mathbb{R}^n$. Then

- Any $k$ linearly independent vectors in $V$ form a basis for $V$.
- Any $k$ vectors in $V$ that span $V$ form a basis.
Midterm 2

Piazza Questions
Matrix Algebra

Solve for $X$:

$$(A - AX)^{-1} = X^{-1}B$$
Matrix Algebra

True/False.

\[(A + B)^2 = A^2 + 2AB = B^2\]

If \(A\) and \(B\) are invertible then \(A + B\) is invertible.

If \(Ax = 0\) has the trivial solution then \(A\) is invertible.
Elementary Matrices

Find a product of elementary matrices that equals

\[
\begin{pmatrix}
2 & 1 \\
1 & 1 \\
\end{pmatrix}
\]
Areas and Determinants

Find the area of the triangle in $\mathbb{R}^2$ with vertices $(-5, 3)$, $(0, 5)$, and $(47, 24)$.

Let $B$ be a ball of radius 1 in $\mathbb{R}^3$. The volume of $B$ is $4\pi/3$. Let

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 9 \\ 0 & 0 & 2 \end{pmatrix}$$

What is the volume of $T_A(B)$?
Inverses and Determinants

Find the matrix of cofactors for

\[ A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \]

What is \( A^{-1} \)?
Properties of Determinants

Suppose that $A$ is a $5 \times 5$ matrix with determinant 3.

Is $A$ invertible?

What is $\det(-A)$?

What is $\det(A^{-1})$?

What is $\det A^T$?

What is the determinant of the matrix obtained from $A$ by replacing the first row with twice the first row plus the second row?
Bases and Coordinates

Let
\[ b_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ and } b_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \]

and \( B = \{b_1, b_2\} \). Explain why \( B \) is a basis for \( \mathbb{R}^2 \).

What is \([e_1]_B\)? In other words, what are the \( B \)-coordinates of \( e_1 = (1, 0) \)?
Computing Determinants

Compute:

$$\det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$