Mathematics 1553
Midterm 2
Prof. Margalit

Section D1/Isabella D2/Kyle D3/Kalen D4/Sidhanth (circle one!)
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1. Answer the following questions. No justification for your answer is required:

- Let $V$ be the set of solutions to $x + y + z = 0$ in $\mathbb{R}^3$. Is $V$ a subspace of $\mathbb{R}^3$?
  
  YES  NO

- Say that $V$ is a plane in $\mathbb{R}^3$, that $v$ and $w$ are two vectors in $V$, and that neither $v$ or $w$ is a multiple of the other. Must it be true that $\{v, w\}$ is a basis for $V$?
  
  YES  NO

- If $A$ is an invertible $n \times n$ matrix, then must it be true that the columns of $A$ form a basis for $\mathbb{R}^n$?
  
  YES  NO

- Suppose $A$ is a $4 \times 4$ matrix and $Ax = e_1$ has infinitely many solutions. Is $A$ invertible?
  
  YES  NO

- Suppose that $A$ is a $2 \times 3$ matrix and that the column space for $A$ is a plane. Is it possible for $Ax = b$ to have infinitely many solutions?
  
  YES  NO
2. Answer the following questions. No justification for your answer is required.

- Complete the definition: **Vectors** $v_1, \ldots, v_k$ **form a basis for a subspace** $V$ of $\mathbb{R}^n$ **if...**

  they are lin. ind. and they span $V$

- Let $V$ be the subset of $\mathbb{R}^2$ given by the first quadrant. In other words:

  $$V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid a \geq 0 \text{ and } b \geq 0 \right\}$$

Which parts of the definition of a subspace are satisfied by $V$? Circle all that apply.

- (a) the zero vector is in $V$
- (b) if $v$ and $w$ are in $V$ then $v + w$ is in $V$
- (c) if $v$ is in $V$ and $c$ is a real number then $cv$ is in $V$
- (d) none of the above

- Solve the following equation for $X$. Assume that all matrices that arise are invertible.

  $$X + AX = B$$

  $$\sim (I + A)X = B \quad \sim X = (I + A)^{-1}B$$

- Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. On the left-hand side, draw the null space of $A$. On the right-hand side, draw the column space of $A$. 

![Graphs showing null space and column space](image-url)
3. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given by clockwise rotation by $\pi/4$ and let $U : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the formula

$$
U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x - y \end{pmatrix}
$$

What is the standard matrix for $T$?

$$
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
$$

What is the standard matrix for $U$?

$$
\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}
$$

What is the range of $U$?

$\mathbb{R}^2$

Which of the following two compositions makes sense? $T \circ U$ $U \circ T$

Write down the standard matrix for the composition you chose.

$$
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}
$$
4. Consider the following matrix $A$ and its reduced row echelon form:

$$A = \begin{pmatrix} 3 & 6 & -1 \\ 4 & 8 & -1 \\ 4 & 8 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $T$ be the matrix transformation $T(v) = Av$.

Find a basis for the null space of $A$.

$$\begin{cases} x + 2y = 0 \\ y = y \\ z = 0 \end{cases} \rightarrow \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Find a basis for column space of $A$.

$$\left\{ \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right\}$$

Are there two different vectors $v$ and $w$ with $T(v) = T(w)$ and $T(v) \neq 0$? $\bigcirc$ Yes $\bigcirc$ No

If you answered yes, find such a $v$ and $w$. If you answered no, explain why not.

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$
5. Find the inverse of the following matrix.

\[
\begin{pmatrix}
3 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 4 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
3 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 4 \\
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 4 \\
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1-30 \\
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -4121 \\
0 & 1 & 1-30 \\
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 1 \\
\end{pmatrix}
\]

Find a matrix \( A \) so the domain of \( T(v) = Av \) is \( \mathbb{R}^3 \) and the range is the line \( y = 2x \) in \( \mathbb{R}^2 \).

\[
\begin{pmatrix}
1 & 0 & 0 \\
2 & 0 & 0 \\
\end{pmatrix}
\]

Find a \( 2 \times 2 \) matrix \( A \) so that \( A \neq I \) and so that \( A^4 = I \). Hint: Find a linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) so that \( T \circ T \circ T \circ T \) is the identity.

\[ T = \text{rotation} \ 2\pi/4 \] 

\[
\begin{pmatrix}
0 & -1 \\
1 & 0 \\
\end{pmatrix}
\]