

Announcements Jan 29

- Midterm 1 on **Friday**
- WeBWorK 2.3, 2.4 & 2.5 due Thursday
- My office hours Monday 3-4 and Wed 2-3
- TA office hours in Skiles 230 (you can go to any of these!)
 - ▶ Isabella Thu 2-3
 - ▶ Kyle Thu 1-3
 - ▶ Kalen Mon/Wed 1-1:50
 - ▶ Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel
- Supplemental problems **and practice exams** on the master web site

Section 2.5

Linear Independence

Section 2.5 Outline

- Understand what is means for a set of vectors to be linearly independent
- Understand how to check if a set of vectors is linearly independent

Linear Independence

Basic question: What is the smallest number of vectors needed in the parametric solution to a linear system?

A set of vectors $\{v_1, \dots, v_k\}$ in \mathbb{R}^n is **linearly independent** if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

has only the trivial solution. It is **linearly dependent** otherwise.

So, linearly dependent means there are x_1, x_2, \dots, x_k not all zero so that

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

This is a *linear dependence* relation.

Linear Independence

A set of vectors $\{v_1, \dots, v_k\}$ in \mathbb{R}^n is **linearly independent** if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

has only the trivial solution.

Fact. The columns of A are linearly independent
 $\Leftrightarrow Ax = 0$ has only the trivial solution.
 $\Leftrightarrow A$ has a pivot in each column

Why?

Linear Independence

Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Linear Independence

When is $\{v\}$ is linearly dependent?

When is $\{v_1, v_2\}$ is linearly dependent?

When is the set $\{v_1, v_2, \dots, v_k\}$ linearly dependent?

Fact. The set $\{v_1, v_2, \dots, v_k\}$ is linearly independent if and only if they span a k -dimensional plane.

Fact. The set $\{v_1, v_2, \dots, v_k\}$ is linearly dependent if and only if we can remove a vector from the set without changing the dimension of the span.

Fact. The set $\{v_1, v_2, \dots, v_k\}$ is linearly dependent if and only if some v_i lies in the span of v_1, \dots, v_{i-1} .

▶ Demo

Span and Linear Independence

Is $\left\{ \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Try using the last Fact.

Linear independence and free variables

Theorem. Let v_1, \dots, v_k be vectors in \mathbb{R}^n and consider the vector equation

$$x_1 v_1 + \dots + x_k v_k = 0.$$

The set of vectors corresponding to non-free variables are linearly independent.

So, given a bunch of vectors v_1, \dots, v_k , if you want to find a collection of v_i that are linearly independent, you put them in the columns of a matrix, row reduce, find the pivots, and then take the **original** v_i corresponding to those columns.

Example. Try this with $(1, 1, 1)$, $(2, 2, 2)$, and $(1, 2, 3)$.

Linear independence and coordinates

Fact. If v_1, \dots, v_k are linearly independent vectors then we can write each element of

$$\text{Span}\{v_1, \dots, v_k\}$$

in exactly one way as a linear combination of v_1, \dots, v_k .

More on this later, when we get to bases.

Span and Linear Independence

Two More Facts

Fact 1. Say v_1, \dots, v_k are in \mathbb{R}^n . If $k > n$, then $\{v_1, \dots, v_k\}$ is linearly dependent.

Fact 2. If one of v_1, \dots, v_k is 0, then $\{v_1, \dots, v_k\}$ is linearly dependent.

Parametric vector form and linear independence

Poll

Say you find the parametric vector form for a homogeneous system of linear equations, and you find that the set of solutions is the span of certain vectors. Then those vectors are...

1. always linearly independent
2. sometimes linearly independent
3. never linearly independent

Example. In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Parametric Vector Forms and Linear Independence

In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

The two vectors that appear are linearly independent (why?). This means that we can't write the solution with fewer than two vectors (why?). This also means that this way of writing the solution set is efficient: for each solution, there is only one choice of x_3 and x_4 that gives that solution.

Summary of Section 2.5

- A set of vectors $\{v_1, \dots, v_k\}$ in \mathbb{R}^n is **linearly independent** if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

has only the trivial solution. It is **linearly dependent** otherwise.

- The cols of A are linearly independent
 - $\Leftrightarrow Ax = 0$ has only the trivial solution.
 - $\Leftrightarrow A$ has a pivot in each column
- The number of pivots of A equals the dimension of the span of the columns of A
- The set $\{v_1, \dots, v_k\}$ is linearly independent \Leftrightarrow they span a k -dimensional plane
- The set $\{v_1, \dots, v_k\}$ is linearly dependent \Leftrightarrow some v_i lies in the span of v_1, \dots, v_{i-1} .
- To find a collection of linearly independent vectors among the $\{v_1, \dots, v_k\}$