Sections 3.1

Matrix Transformations

Section 3.1 Outline

- Learn to think of matrices as functions, called matrix transformations
- Learn the associated terminology: domain, codomain, range
- Understand what certain matrices do to $\mathbb{R}^n$

From matrices to functions

Let A be an $m \times n$ matrix.

We define a function

$$T : \mathbb{R}^n \to \mathbb{R}^m$$
$$T(v) = Av$$

This is called a matrix transformation.

The domain of T is $\mathbb{R}^n$.

The co-domain of T is $\mathbb{R}^m$.

The range of T is the set of outputs: $\operatorname{Col}(A)$

This gives us a*nother* point of view of $Ax = b$

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$.

What is $T(u)$?

Find v in $\mathbb{R}^2$ so that $T(v) = b$

Find a vector in $\mathbb{R}^3$ that is not in the range of T.

For a square matrix we can think of the associated matrix transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

as doing something to $\mathbb{R}^n$.

Example. The matrix transformation T for

$$\left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right)$$

What does T do to $\mathbb{R}^2$?

Square matrices

What does each matrix do to $\mathbb{R}^2$?

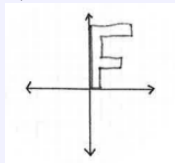$$\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$$

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{cc} 3 & 0 \\ 0 & 3 \end{array} \right)$$

What is the range in each case?

Poll

What does $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ do to this letter F?

Square matrices

What does each matrix do to $\mathbb{R}^2$?

Hint: if you can't see it all at once, see what happens to the x- and y-axes.

$$\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$$

$$\left(\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right)$$

$$\left(\begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right)$$

Examples in $\mathbb{R}^3$

What does each matrix do to $\mathbb{R}^3$?

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

- If A is an $m \times n$ matrix, then the associated matrix transformation T is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\operatorname{Col}(A)$.

- If A is $n \times n$ then T does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation