Announcements Feb 12

- Midterm 2 on March 6
- WeBWorK 2.7+2.9, 3.1 due Thursday
- My office hours Monday 3-4 and Wed 2-3
- Pop-up office hours Wed 3-3:30
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel (different this week)
- Supplemental problems and practice exams on the master web site

Sections 3.2

One-to-one and onto transformations

Section 3.2 Outline

- Learn the definitions of one-to-one and onto functions
- Determine if a given matrix transformation is one-to-one and/or onto

One-to-one

$T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

In other words: different inputs have different outputs.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is one-to-one
- the columns of A are linearly independent
- $Ax = 0$ has only the trivial solution
- A has a pivot in each column
- the range of T has dimension n

What can we say about the relative sizes of m and n if T is one-to-one?

Draw a picture of the range of a one-to-one matrix transformation $\mathbb{R} \to \mathbb{R}^3$.

Onto

$T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is onto
- the columns of A span $\mathbb{R}^m$
- A has a pivot in each row
- $Ax = b$ is consistent for all b in $\mathbb{R}^m$
- the range of T has dimension m

What can we say about the relative sizes of m and n if T is onto?

Give an example of an onto matrix transformation $\mathbb{R}^3 \to \mathbb{R}$.

One-to-one and Onto

Do the following give matrix transformations that are one-to-one? onto?

$$\begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

One-to-one and Onto

Which of the previously-studied matrix transformations of $\mathbb{R}^2$ are one-to-one? Onto?

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ reflection

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projection

$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ scaling

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ shear

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotation

Which are one to one / onto?

Poll

Which are one to one / onto?

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

▸ Demo

▸ Demo

▸ Demo

Summary of Section 3.2

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is one-to-one
  - the columns of A are linearly independent
  - $Ax = 0$ has only the trivial solution
  - A has a pivot in each column
  - the range has dimension n
- $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is onto
  - the columns of A span $\mathbb{R}^m$
  - A has a pivot in each row
  - $Ax = b$ is consistent for all b in $\mathbb{R}^m$.
  - the range of T has dimension m