Announcements Feb 19

• Midterm 2 on March 6
• WeBWorK 2.7+2.9, 3.1 due Thursday
• Mid-semester evaluation under Quizzes on Canvas (due today)
• My office hours Monday 3-4 and Wed 2-3 in Skiles 234
• Pop-up office hours Wed 11-11:30 this week in Skiles 234
• TA office hours in Skiles 230 (you can go to any of these!)
  ▶ Isabella Thu 2-3
  ▶ Kyle Thu 1-3
  ▶ Kalen Mon/Wed 1-1:50
  ▶ Sidhanth Tue 10:45-11:45

• PLUS sessions Mon/Wed 6-7 LLC West with Miguel
• Supplemental problems and practice exams on the master web site
Section 3.4
Matrix Multiplication
Section 3.4 Outline

- Understand composition of linear transformations
- Learn how to multiply matrices
- Learn the connection between these two things
Function composition

Remember from calculus that if $f$ and $g$ are functions then the composition $f \circ g$ is a new function defined as follows:

$$f \circ g(x) = f(g(x))$$

In words: first apply $g$, then $f$.

Example: $f(x) = x^2$ and $g(x) = x + 1$.

Note that $f \circ g$ is usually different from $g \circ f$. 
Composition of linear transformations

We can do the same thing with linear transformations \( T : \mathbb{R}^p \to \mathbb{R}^m \) and \( U : \mathbb{R}^n \to \mathbb{R}^p \) and make the composition \( T \circ U \).

Notice that both have an \( p \). Why?

What are the domain and codomain for \( T \circ U \)?

Natural question: What is the matrix for \( T \circ U \)?

Associative property: \( (S \circ T) \circ U = S \circ (T \circ U) \)

Why?
Composition of linear transformations

Example. $T = \text{projection to } y\text{-axis}$ and $U = \text{reflection about } y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?

What about $U \circ T$?
Suppose $A$ is an $m \times n$ matrix. We write $a_{ij}$ or $A_{ij}$ for the $ij$th entry.

If $A$ is $m \times n$ and $B$ is $n \times p$, then $AB$ is $m \times p$ and

$$(AB)_{ij} = r_i \cdot b_j$$

where $r_i$ is the $i$th row of $A$, and $b_j$ is the $j$th column of $B$.

Or: the $j$th column of $AB$ is $A$ times the $j$th column of $B$.

Multiply these matrices (both ways):

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{pmatrix}$$
Matrix Multiplication and Linear Transformations

As above, the composition \( T \circ U \) means: do \( U \) then do \( T \)

**Fact.** Suppose that \( A \) and \( B \) are the standard matrices for the linear transformations \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) and \( U : \mathbb{R}^p \rightarrow \mathbb{R}^n \). The standard matrix for \( T \circ U \) is \( AB \).

Why?

\[
(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv)
\]

So we need to check that \( A(Bv) = (AB)v \). Enough to do this for \( v = e_i \). In this case \( Bv \) is the \( i \)th column of \( B \). So the left-hand side is \( A \) times the \( i \)th column of \( B \). The right-hand side is the \( i \)th column of \( AB \) which we already said was \( A \) times the \( i \)th column of \( B \). It works!
Matrix Multiplication and Linear Transformations

**Fact.** Suppose that $A$ and $B$ are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is $AB$.

**Example.** $T =$ projection to $y$-axis and $U =$ reflection about $y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of \( \mathbb{R}^3 \) that reflects through the \( xy \)-plane and then projects onto the \( yz \)-plane.
Discussion Question

Are there nonzero matrices $A$ and $B$ with $AB = 0$?

1. Yes
2. No
Properties of Matrix Multiplication

- \( A(BC) = (AB)C \)
- \( A(B + C) = AB + AC \)
- \( (B + C)A = BA + CA \)
- \( r(AB) = (rA)B = A(rB) \)
- \( (AB)^T = B^T A^T \)
- \( I_mA = A = AI_n \), where \( I_k \) is the \( k \times k \) identity matrix.

Multiplication is associative because function composition is (this would be hard to check from the definition!).

Warning!

- \( AB \) is not always equal to \( BA \)
- \( AB = AC \) does not mean that \( B = C \)
- \( AB = 0 \) does not mean that \( A \) or \( B \) is 0
Sums and Scalar Multiples

Same as for vectors: component-wise, so matrices must be same size to add.

\[ A + B = B + A \]

\[ (A + B) + C = A + (B + C) \]

\[ r(A + B) = rA + rB \]

\[ (r + s)A = rA + sA \]

\[ (rs)A = r(sA) \]

\[ A + 0 = A \]

(We can define linear transformations \( T + U \) ad \( cT \), and so all of the above facts are also facts about linear transformations.)
Summary of Section 3.4

- **Composition**: \((T \circ U)(v) = T(U(v))\) \(\text{ (do } U \text{ then } T\)\)
- **Matrix multiplication**: \((AB)_{ij} = r_i \cdot b_j\)
- **Matrix multiplication**: the \(i\)th column of \(AB\) is \(A(b_i)\)
- Suppose that \(A\) and \(B\) are the standard matrices for the linear transformations \(T : \mathbb{R}^n \to \mathbb{R}^m\) and \(U : \mathbb{R}^p \to \mathbb{R}^n\). The standard matrix for \(T \circ U\) is \(AB\).
- **Warning**!
  - \(AB\) is not always equal to \(BA\)
  - \(AB = AC\) does not mean that \(B = C\)
  - \(AB = 0\) does not mean that \(A\) or \(B\) is 0