Announcements Feb 19

- Midterm 2 on March 6
- WeBWorK 2.7+2.9, 3.1 due Thursday
- Mid-semester evaluation under Quizzes on Canvas (due today)
- My office hours Monday 3-4 and Wed 2-3 in Skiles 234
- Pop-up office hours Wed 11-11:30 this week in Skiles 234
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel
- Supplemental problems and practice exams on the master web site

Section 3.4

Matrix Multiplication

Section 3.4 Outline

- Understand composition of linear transformations
- Learn how to multiply matrices
- Learn the connection between these two things

Function composition

Remember from calculus that if f and g are functions then the composition $f \circ g$ is a new function defined as follows:

$$f \circ g(x) = f(g(x))$$

In words: first apply g, then f.

Example: $f(x) = x^2$ and $g(x) = x + 1$.

Note that $f \circ g$ is usually different from $g \circ f$.

Composition of linear transformations

We can do the same thing with linear transformations $T : \mathbb{R}^p \to \mathbb{R}^m$ and $U : \mathbb{R}^n \to \mathbb{R}^p$ and make the composition $T \circ U$.

Notice that both have an p. Why?

What are the domain and codomain for $T \circ U$?

Natural question: What is the matrix for $T \circ U$?

Associative property: $(S \circ T) \circ U = S \circ (T \circ U)$

Why?

Composition of linear transformations

Example. $T =$ projection to y-axis and
$U =$ reflection about $y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?

What about $U \circ T$?

Matrix Multiplication

And now for something completely different (not really!)

Suppose A is an $m \times n$ matrix. We write a_{ij} or A_{ij} for the ijth entry.

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$ and

$$(AB)_{ij} = r_i \cdot b_j$$

where r_i is the ith row of A, and b_j is the jth column of B.

Or: the jth column of AB is A times the jth column of B.

Multiply these matrices (both ways):

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{pmatrix}$$

Matrix Multiplication and Linear Transformations

As above, the composition $T \circ U$ means: do U then do T

Fact. Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.

Why?

$$(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv)$$

So we need to check that $A(Bv) = (AB)v$. Enough to do this for $v = e_i$. In this case Bv is the ith column of B. So the left-hand side is A times the ith column of B. The right-hand side is the ith column of AB which we already said was A times the ith column of B. It works!

Matrix Multiplication and Linear Transformations

Fact. Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.

Example. $T =$ projection to y-axis and $U =$ reflection about $y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane.

Discussion Question

Are there nonzero matrices A and B with $AB = 0$?

1. Yes
2. No

Properties of Matrix Multiplication

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $r(AB) = (rA)B = A(rB)$
- $(AB)^T = B^T A^T$
- $I_m A = A = AI_n$, where I_k is the $k \times k$ identity matrix.

Multiplication is associative because function composition is (this would be hard to check from the definition!).

Warning!

- AB is not always equal to BA
- $AB = AC$ does not mean that $B = C$
- $AB = 0$ does not mean that A or B is 0

Sums and Scalar Multiples

Same as for vectors: component-wise, so matrices must be same size to add.

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$(rs)A = r(sA)$$

$$A + 0 = A$$

(We can define linear transformations $T + U$ ad cT, and so all of the above facts are also facts about linear transformations.)

Summary of Section 3.4

- Composition: $(T \circ U)(v) = T(U(v))$ (do U then T)
- Matrix multiplication: $(AB)_{ij} = r_i \cdot b_j$
- Matrix multiplication: the ith column of AB is $A(b_i)$
- Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.
- Warning!
  - AB is not always equal to BA
  - $AB = AC$ does not mean that $B = C$
  - $AB = 0$ does not mean that A or B is 0