Section 3.6

The invertible matrix theorem
Section 3.6 Outline

- The invertible matrix theorem
The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

1. $A$ is invertible
2. $T$ is invertible
3. The reduced row echelon form of $A$ is $I_n$
4. $A$ has $n$ pivots
5. $Ax = 0$ has only 0 solution
6. $\text{Nul}(A) = \{0\}$
7. $\text{nullity}(A) = 0$
8. columns of $A$ are linearly independent
9. columns of $A$ form a basis for $\mathbb{R}^n$
10. $T$ is one-to-one
11. $Ax = b$ is consistent for all $b$ in $\mathbb{R}^n$
12. $Ax = b$ has a unique solution for all $b$ in $\mathbb{R}^n$
13. columns of $A$ span $\mathbb{R}^n$
14. $\text{Col}(A) = \mathbb{R}^n$
15. $\text{rank}(A) = n$
16. $T$ is onto
17. $A$ has a left inverse
18. $A$ has a right inverse
The Invertible Matrix Theorem

There are two kinds of square matrices, invertible and non-invertible matrices.

For invertible matrices, all of the conditions in the IMT hold. And for a non-invertible matrix, all of them fail to hold.

One way to think about the theorem is: there are lots of conditions equivalent to a matrix having a pivot in every row, and lots of conditions equivalent to a matrix having a pivot in every column, and when the matrix is a square, all of these many conditions become equivalent.
Example

Determine whether $A$ is invertible. $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$

It isn't necessary to find the inverse. Instead, we may use the Invertible Matrix Theorem by checking whether we can row reduce to obtain three pivot columns, or three pivot positions.

\[
A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}
\]

There are three pivot positions, so $A$ is invertible by the IMT (statement c).
The Invertible Matrix Theorem

Poll

Which are true? Why?

m) If $A$ is invertible then the rows of $A$ span $\mathbb{R}^n$

n) If $Ax = b$ has exactly one solution for all $b$ in $\mathbb{R}^n$ then $A$ is row equivalent to the identity.

o) If $A$ is invertible then $A^2$ is invertible

p) If $A^2$ is invertible then $A$ is invertible
Some sample Yes/No questions

In all questions, suppose that $A$ is an $n \times n$ matrix and that $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation.

(1) Suppose that the reduced row echelon form of $A$ does not have any zero rows. Must it be true that $Ax = b$ is consistent for all $b$ in $\mathbb{R}^n$?

   YES                           NO

(2) Suppose that $T$ is one-to-one. Is it possible that the columns of $A$ add up to zero?

   YES                           NO

(3) Suppose that $Ax = e_1$ is not consistent. Is it possible that $T$ is onto?

   YES                           NO
Summary of Section 3.6

- Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.
  
  (1) $A$ is invertible  
  (2) $T$ is invertible  
  (3) The reduced row echelon form of $A$ is $I_n$  
  (4) etc.