Chapter 5

Determinants

Section 4.2

Cofactor expansions

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Outline of Section 4.2

• We will give a recursive formula for the determinant of a square matrix.

We will give a recursive formula.

First some terminology:

 $A_{ij}=ij{\rm th}$ minor of A $A_{ij}=(n-1)\times(n-1)$ matrix obtained by deleting the $i{\rm th}$ row and $j{\rm th}$ column

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

= ijth cofactor of A

Finally:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

Or:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \dots \pm a_{1n}(\det(A_{1n}))$$

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For the recursive formula:

 $\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \dots \pm a_{1n}(\det(A_{1n}))$

Need to start somewhere...

 $1\times 1~\mathrm{matrices}$

 $\det(a_{11}) = a_{11}$

 $2\times 2~\mathrm{matrices}$

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}C_{11} + a_{12}C_{12}$$
$$= a_{11}\det(A_{11}) + a_{12}(-\det(A_{12}))$$
$$= a_{11}(a_{22}) + a_{12}(-a_{21})$$

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 $3\times 3~\mathrm{matrices}$

$$\det \left(\begin{array}{rrrr} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right) = \cdots$$

You can write this out. And it is a good exercise. But you won't want to memorize it.

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Determinants

Compute

$$\det \left(\begin{array}{rrrr} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{array} \right)$$

Another formula for 3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

 $-a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$

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Use this formula to compute

$$\det \left(\begin{array}{rrrr} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{array} \right)$$

Expanding across other rows and columns

The formula we gave for det(A) is the expansion across the first row. It turns out you can compute the determinant by expanding across any row or column:

$$det(A) = a_{i1}C_{i1} + \dots + a_{in}C_{in} \text{ for any fixed } i$$
$$det(A) = a_{1j}C_{1j} + \dots + a_{nj}C_{nj} \text{ for any fixed } j$$

Or:

$$\det(A) = a_{i1}(\det(A_{i1})) - a_{i2}(\det(A_{i2})) + \dots \pm a_{in}(\det(A_{in}))$$
$$\det(A) = a_{1j}(\det(A_{1j})) - a_{2j}(\det(A_{2j})) + \dots \pm a_{nj}(\det(A_{nj}))$$

Compute:

$$\det \left(\begin{array}{rrr} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{array} \right)$$

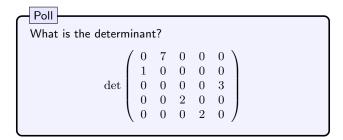
Determinants of triangular matrices

If A is upper (or lower) triangular, det(A) is easy to compute:

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$$\det \left(\begin{array}{rrrr} 2 & 1 & 5 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 10 \end{array}\right)$$

Determinants



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A formula for the inverse

(from Section 3.3)

 $2\times 2~\mathrm{matrices}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

 $n \times n$ matrices

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{pmatrix}$$
$$= \frac{1}{\det(A)} (C_{ij})^T$$

Check that these agree!

The proof uses Cramer's rule (see the notes on the course home page. We're not testing on this - it's just for your information.)

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Summary of Section 4.2

• There is a recursive formula for the determinant of a square matrix:

 $\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \dots \pm a_{1n}(\det(A_{1n}))$

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- We can use the same formula along any row/column.
- There are special formulas for the 2×2 and 3×3 cases.