Chapter 5

Determinants
Section 4.2

Cofactor expansions
Outline of Section 4.2

- We will give a recursive formula for the determinant of a square matrix.
A formula for the determinant

We will give a recursive formula.

First some terminology:

\[ A_{ij} = ij\text{th minor of } A \]
\[ A_{ij} = (n - 1) \times (n - 1) \text{ matrix obtained by deleting the } i\text{th row and } j\text{th column} \]

\[ C_{ij} = (-1)^{i+j} \det(A_{ij}) \]
\[ = ij\text{th cofactor of } A \]

Finally:

\[ \det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \]

Or:

\[ \det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n})) \]
A formula for the determinant

For the recursive formula:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

Need to start somewhere...

1 × 1 matrices

$$\det(a_{11}) = a_{11}$$

2 × 2 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}C_{11} + a_{12}C_{12}$$
$$= a_{11} \det(A_{11}) + a_{12}( - \det(A_{12}))$$
$$= a_{11}(a_{22}) + a_{12}( - a_{21})$$
A formula for the determinant

$3 \times 3$ matrices

$$\text{det} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \cdots$$

You can write this out. And it is a good exercise. But you won’t want to memorize it.
Determinants

Compute

\[
\begin{vmatrix}
5 & 1 & 0 \\
-1 & 3 & 2 \\
4 & 0 & -1
\end{vmatrix}
\]
A formula for the determinant

Another formula for \(3 \times 3\) matrices

\[
\text{det} \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}
\]

Use this formula to compute

\[
\text{det} \begin{pmatrix}
5 & 1 & 0 \\
-1 & 3 & 2 \\
4 & 0 & -1
\end{pmatrix}
\]
Expanding across other rows and columns

The formula we gave for $\det(A)$ is the **expansion across the first row**. It turns out you can compute the determinant by expanding across any row or column:

\[
\det(A) = a_{i1} C_{i1} + \cdots + a_{in} C_{in} \quad \text{for any fixed } i
\]
\[
\det(A) = a_{1j} C_{1j} + \cdots + a_{nj} C_{nj} \quad \text{for any fixed } j
\]

Or:

\[
\det(A) = a_{i1}(\det(A_{i1})) - a_{i2}(\det(A_{i2})) + \cdots \pm a_{in}(\det(A_{in}))
\]
\[
\det(A) = a_{1j}(\det(A_{1j})) - a_{2j}(\det(A_{2j})) + \cdots \pm a_{nj}(\det(A_{nj}))
\]

Compute:

\[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 1 & 0 \\
5 & 9 & 1
\end{pmatrix}
\]
Determinants of triangular matrices

If $A$ is upper (or lower) triangular, $\det(A)$ is easy to compute:

\[
\begin{vmatrix}
2 & 1 & 5 & -2 \\
0 & 1 & 2 & -3 \\
0 & 0 & 5 & 9 \\
0 & 0 & 0 & 10
\end{vmatrix}
\]
Determinants

What is the determinant?

\[
\begin{bmatrix}
0 & 7 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
\end{bmatrix}
\]
A formula for the inverse
(from Section 3.3)

$2 \times 2$ matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$n \times n$ matrices

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{pmatrix} = \frac{1}{\det(A)} (C_{ij})^T$$

Check that these agree!

The proof uses Cramer’s rule (see the notes on the course home page. We’re not testing on this - it’s just for your information.)
Summary of Section 4.2

- There is a recursive formula for the determinant of a square matrix:
  \[ \det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n})) \]
- We can use the same formula along any row/column.
- There are special formulas for the $2 \times 2$ and $3 \times 3$ cases.