Announcements April 1

- Class participation (Piazza polls) is optional for the rest of the semester.
- We will use Blue Jeans Meetings for the rest of the semester.
- The new schedule is on the web page.
- Midterm 3 on April 17
- WeBWorK 5.1 due tomorrow: Thu April 2.
- Official quiz on Friday on Canvas on 4.1, 4.2, 4.3, 5.1.
  It will be open all day Friday, but there will be a time limit.
- My office hours Monday 3-4 and Wed 2-3 on Blue Jeans
- TA office hours on Blue Jeans (you can go to any of these!)
  - Isabella Mon 11-12, Wed 11-12
  - Kyle Wed 3-5, Thu 1-3
  - Kalen Mon/Wed 1-2
  - Sidhanth Tue 10-12
- Supplemental problems and practice exams on the master web site
- Counseling Center: http://counseling.gatech.edu Click

Section 5.2

The characteristic polynomial

Outline of Section 5.2

- How to find the eigenvalues, via the characteristic polynomial
- Techniques for the 3×3 case

Characteristic polynomial

Recall:

$$\lambda \text{ is an eigenvalue of } A \iff A - \lambda I \text{ is not invertible}$$

So to find eigenvalues of A we solve

$$\det(A - \lambda I) = 0$$

The left hand side is a polynomial, the characteristic polynomial of A.

The roots of the characteristic polynomial are the eigenvalues of A.

The eigenrecipe

Say you are given an square matrix A.

Step 1. Find the eigenvalues of A by solving

$$\det(A - \lambda I) = 0$$

Step 2. For each eigenvalue λ_i the λ_i-eigenspace is the solution to

$$(A - \lambda_i I)x = 0$$

To find a basis, find the vector parametric solution, as usual.

Characteristic polynomial

Find the characteristic polynomial and eigenvalues of

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Characteristic polynomials, trace, and determinant

The trace of a matrix is the sum of the diagonal entries.

The characteristic polynomial of an $n \times n$ matrix A is a polynomial with leading term $(-1)^n$, next term $(-1)^{n-1}\mathrm{trace}(A)$, and constant term $\det(A)$:

$$(-1)^n\lambda^n + (-1)^{n-1}\mathrm{trace}(A)\lambda^{n-1} + \cdots + \det(A)$$

So for a 2×2 matrix:

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A)$$

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ -3 & 0 & -1 \end{pmatrix}$$

What are the eigenvalues? Hint: Don't multiply everything out!

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ 4 & 2 & 0 \end{pmatrix}$$

Answer: $-\lambda^3 + 9\lambda^2 - 8\lambda$

What are the eigenvalues?

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Answer:

$$-\lambda^3 + 3\lambda + 2$$

What are the eigenvalues?

Hint: We already know one eigenvalue! Polynomial long division $\rightsquigarrow$

$$(\lambda - 2)(-\lambda^2 - 2\lambda - 1)$$

Don't really need long division: the first and last terms of the quadratic are easy to find; can guess and check the other term.

Without the hint, could use the rational root theorem: any integer root of a polynomial with leading coefficient ± 1 divides the constant term.

Eigenvalues

Triangular matrices

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

Warning! You cannot find eigenvalues by row reducing and then using this fact. You need to work with the original matrix. Finding eigenspaces involves row reducing $A - \lambda I$, but there is no row reduction in finding eigenvalues.

Eigenvalues

Geometrically defined matrices

Say that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation that projects onto the plane $2x + 3y = 0$ and that A is the standard matrix for T. What are the eigenvalues of A?

Say that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation that projects onto the plane $2x + 3y - z = 0$ and that A is the standard matrix for T. What are the eigenvalues of A?

Algebraic multiplicity

The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example. Find the algebraic multiplicities of the eigenvalues for

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Fact. The sum of the algebraic multiplicities of the (real) eigenvalues of an $n \times n$ matrix is at most n.

Review of Section 5.2

True or false: every $n \times n$ matrix has an eigenvalue.

True or false: every $n \times n$ matrix has n distinct eigenvalues.

True or false: the nullity of $A - \lambda I$ is the dimension of the λ-eigenspace.

What are the eigenvalues for the standard matrix for a reflection?

Summary of Section 5.2

- The characteristic polynomial of A is $\det(A - \lambda I)$
- The roots of the characteristic polynomial for A are the eigenvalues
- Techniques for 3×3 matrices:
  - Don't multiply out if there is a common factor
  - If there is no constant term then factor out λ
  - If the matrix is triangular, the eigenvalues are the diagonal entries
  - Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - Use the geometry to determine an eigenvalue
- Given an square matrix A:
  - The eigenvalues are the solutions to $\det(A - \lambda I) = 0$
  - Each λ_i-eigenspace is the solution to $(A - \lambda_i I)x = 0$