Announcements April 1

- Class participation (Piazza polls) is optional for the rest of the semester.
- We will use Blue Jeans Meetings for the rest of the semester.
- The new schedule is on the web page.
- Midterm 3 on April 17
- WeBWorK 5.1 due tomorrow: Thu April 2.
- Official quiz on Friday on Canvas on 4.1, 4.2, 4.3, 5.1. It will be open all day Friday, but there will be a time limit.
- My office hours Monday 3-4 and Wed 2-3 on Blue Jeans
- TA office hours on Blue Jeans (you can go to any of these!)
  - Isabella Mon 11-12, Wed 11-12
  - Kyle Wed 3-5, Thu 1-3
  - Kalen Mon/Wed 1-2
  - Sidhanth Tue 10-12
- Supplemental problems and practice exams on the master web site
- Counseling Center: http://counseling.gatech.edu
Section 5.2

The characteristic polynomial
Outline of Section 5.2

- How to find the eigenvalues, via the characteristic polynomial
- Techniques for the $3 \times 3$ case
Characteristic polynomial

Recall:

\[ \lambda \text{ is an eigenvalue of } A \iff A - \lambda I \text{ is not invertible} \]

So to find eigenvalues of \( A \) we solve

\[ \det(A - \lambda I) = 0 \]

The left hand side is a polynomial, the characteristic polynomial of \( A \).

The roots of the characteristic polynomial are the eigenvalues of \( A \).
The eigenrecipe

Say you are given an square matrix $A$.

**Step 1.** Find the eigenvalues of $A$ by solving

$$\det(A - \lambda I) = 0$$

**Step 2.** For each eigenvalue $\lambda_i$ the $\lambda_i$-eigenspace is the solution to

$$(A - \lambda_i I)x = 0$$

To find a basis, find the vector parametric solution, as usual.
Characteristic polynomial

Find the characteristic polynomial and eigenvalues of

\[
\begin{pmatrix}
5 & 2 \\
2 & 1
\end{pmatrix}
\]
**Characteristic polynomials, trace, and determinant**

The **trace** of a matrix is the sum of the diagonal entries.

The characteristic polynomial of an $n \times n$ matrix $A$ is a polynomial with leading term $(-1)^n$, next term $(-1)^{n-1}\text{trace}(A)$, and constant term $\text{det}(A)$:

$$(-1)^n \lambda^n + (-1)^{n-1}\text{trace}(A)\lambda^{n-1} + \cdots + \text{det}(A)$$

So for a $2 \times 2$ matrix:

$$\lambda^2 - \text{trace}(A)\lambda + \text{det}(A)$$
Find the characteristic polynomial of the following matrix.

\[
\begin{pmatrix}
7 & 0 & 3 \\
-3 & 2 & -3 \\
-3 & 0 & -1
\end{pmatrix}
\]

What are the eigenvalues? Hint: Don't multiply everything out!
Characteristic polynomials

$3 \times 3$ matrices

Find the characteristic polynomial of the following matrix.

\[
\begin{pmatrix}
7 & 0 & 3 \\
-3 & 2 & -3 \\
4 & 2 & 0
\end{pmatrix}
\]

Answer: $-\lambda^3 + 9\lambda^2 - 8\lambda$

What are the eigenvalues?
Find the characteristic polynomial of the rabbit population matrix.

\[
\begin{pmatrix}
0 & 6 & 8 \\
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0
\end{pmatrix}
\]

Answer:

\[-\lambda^3 + 3\lambda + 2\]

What are the eigenvalues?

*Hint:* We already know one eigenvalue! Polynomial long division \(\Rightarrow\)

\[(\lambda - 2)(-\lambda^2 - 2\lambda - 1)\]

Don’t really need long division: the first and last terms of the quadratic are easy to find; can guess and check the other term.

Without the hint, could use the rational root theorem: any integer root of a polynomial with leading coefficient \(\pm 1\) divides the constant term.
Eigenvalues
Triangular matrices

**Fact.** The eigenvalues of a triangular matrix are the diagonal entries.

*Why?*

**Warning!** You cannot find eigenvalues by row reducing and then using this fact. You need to work with the original matrix. Finding eigenspaces involves row reducing $A - \lambda I$, but there is no row reduction in finding eigenvalues.
Eigenvalues
Geometrically defined matrices

Say that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation that projects onto the plane $2x + 3y = 0$ and that $A$ is the standard matrix for $T$. What are the eigenvalues of $A$?

Say that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation that projects onto the plane $2x + 3y - z = 0$ and that $A$ is the standard matrix for $T$. What are the eigenvalues of $A$?
Algebraic multiplicity

The algebraic multiplicity of an eigenvalue $\lambda$ is its multiplicity as a root of the characteristic polynomial.

*Example.* Find the algebraic multiplicities of the eigenvalues for

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

*Fact.* The sum of the algebraic multiplicities of the (real) eigenvalues of an $n \times n$ matrix is at most $n$. 
Review of Section 5.2

True or false: every $n \times n$ matrix has an eigenvalue.

True or false: every $n \times n$ matrix has $n$ distinct eigenvalues.

True or false: the nullity of $A - \lambda I$ is the dimension of the $\lambda$-eigenspace.

What are the eigenvalues for the standard matrix for a reflection?
Summary of Section 5.2

- The characteristic polynomial of $A$ is $\det(A - \lambda I)$
- The roots of the characteristic polynomial for $A$ are the eigenvalues
- Techniques for $3 \times 3$ matrices:
  - Don’t multiply out if there is a common factor
  - If there is no constant term then factor out $\lambda$
  - If the matrix is triangular, the eigenvalues are the diagonal entries
  - Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - Use the geometry to determine an eigenvalue
- Given an square matrix $A$:
  - The eigenvalues are the solutions to $\det(A - \lambda I) = 0$
  - Each $\lambda_i$-eigenspace is the solution to $(A - \lambda_i I)x = 0$