Eigenvalues in Structural Engineering

Watch this video about the Tacoma Narrows bridge. ▸ Watch

Here are some toy models. ▸ Check it out

The masses move the most at their natural frequencies ω. To find those, use the spring equation: $mx'' = -kx \quad \rightsquigarrow \quad \sin(\omega t)$.

With 3 springs and 2 equal masses, we get:

$$\begin{aligned} mx_1'' &= -kx_1 + k(x_2 - x_1) \\ mx_2'' &= -kx_2 + k(x_1 - x_2) \end{aligned}$$

Guess a solution $x_1(t) = A_1(\cos(\omega t) + i\sin(\omega t))$ and similar for x_2. Finding ω reduces to finding eigenvalues of $\begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix}$.

Eigenvectors: $(1,1)$ & $(1,-1)$ (in/out of phase) ▸ Details

Section 5.4

Diagonalization

Section 5.4 Outline

- Diagonalization
- Using diagonalization to take powers
- Algebraic versus geometric dimension

We understand diagonal matrices

We completely understand what diagonal matrices do to $\mathbb{R}^n$. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

We have seen that it is useful to take powers of matrices: for instance in computing rabbit populations.

If A is diagonal, powers of A are easy to compute. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{10} =$$

Powers of matrices that are similar to diagonal ones

What if A is not diagonal? Suppose want to understand the matrix

$$A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$$

geometrically? Or take it's 10th power? What would we do?

What if I give you the following equality:

$$\begin{array}{ccccc} \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} & = & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \\ A & = & C & D & C^{-1} \end{array}$$

This is called diagonalization.

How does this help us understand A? Or find A^{10}?

Powers of matrices that are similar to diagonal ones

What if I give you the following equality:

$$\begin{array}{ccccccc} \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} & = & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \\ A & = & C & D & C^{-1} \end{array}$$

This is called diagonalization.

How does this help us understand A? Or find A^{10}? Demo

Diagonalization

Suppose A is $n \times n$. We say that A is diagonalizable if we can write:

$$A = CDC^{-1} \qquad\qquad D = \text{diagonal}$$

We say that A is similar to D.

How does this factorization of A help describe what A does to $\mathbb{R}^n$?
How does this help us take powers of A?

Understanding the rabbit example: since 2 is the largest eigenvalue, (almost) all other vectors get pulled towards that eigenvector. Compare with the example from the last slide.

Diagonalization

The recipe

Theorem. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

In this case

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}$$

$$= \qquad C \qquad D \qquad C^{-1}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues (in order).

Why?

Example

Diagonalize if possible.

$$\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$$

Example

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Example

Diagonalize if possible.

$$\begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$$

Demo

More Examples

Diagonalize if possible.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Poll

Poll

Which are diagonalizable?

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Distinct Eigenvalues

Fact. If A has n distinct eigenvalues, then A is diagonalizable.

Why?

Non-Distinct Eigenvalues

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity")

Then

1. $d_i \leq a_i$ for all i
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n$

   $\Leftrightarrow \Sigma a_i = n$ and $d_i = a_i$ for all i

So: if you find one eigenvalue where the geometric multiplicity is less than the algebraic multiplicity, the matrix is not diagonalizable.

Review of Section 5.4

True or false: If A is a 3×3 matrix with eigenvalues 0, 1, and 2, then A is diagonalizable.

True or false: It is possible for an eigenspace to be 0-dimensional.

Summary of Section 5.4

- A is diagonalizable if $A = CDC^{-1}$ where D is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $\Leftrightarrow$ the sum of the geometric dimensions of the eigenspaces in n
- If A has n distinct eigenvalues it is diagonalizable