Section 6.2
Orthogonal complements
Outline of Section 6.2

- Orthogonal complements
- Computing orthogonal complements
Orthogonal complements

\[ W = \text{subspace of } \mathbb{R}^n \]
\[ W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w \text{ for all } w \in W \} \]

**Question.** What is the orthogonal complement of a line in \( \mathbb{R}^3 \)?

**Facts.**

1. \( W^\perp \) is a subspace of \( \mathbb{R}^n \)
2. \((W^\perp)^\perp = W\)
3. \(\dim W + \dim W^\perp = n\)
4. If \( W = \text{Span}\{w_1, \ldots, w_k\} \) then \( W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w_i \text{ for all } i \}\)
5. The intersection of \( W \) and \( W^\perp \) is \( \{0\} \).
Orthogonal complements
Finding them

Problem. Let $W = \text{Span}\{(1, 1, -1)\}$. Find the equation of the plane $W^\perp$. 
Orthogonal complements
Finding them

Problem. Let $W = \text{Span}\{(1, 1, -1), (-1, 2, 1)\}$. Find a system of equations describing the line $W^\perp$. 
Orthogonal complements
Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for $W$, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \iff x$ is orthogonal to each row of $A$
Orthogonal complements
Finding them

Problem. Let $W = \text{Span}\{(1, 1, -1), (-1, 2, 1)\}$. Find a system of equations describing the line $W^\perp$.

Why? $Ax = 0 \iff x$ is orthogonal to each row of $A$

Theorem. $A = m \times n$ matrix

$$(\text{Row}A)^\perp = \text{Nul} A$$

Geometry $\leftrightarrow$ Algebra
Orthogonal decomposition

Fact. Say $W$ is a subspace of $\mathbb{R}^n$. Then any vector $v$ in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where $v_W$ is in $W$ and $v_{W^\perp}$ is in $W^\perp$.

Why? Say that $w_1 + w_1' = w_2 + w_2'$ where $w_1$ and $w_2$ are in $W$ and $w_1'$ and $w_2'$ are in $W^\perp$. Then $w_1 - w_2 = w_2' - w_1'$. But the former is in $W$ and the latter is in $W^\perp$, so they must both be equal to 0.

Next time: Find $v_W$ and $v_{W^\perp}$.
Orthogonal Projections

Many applications, including:
Review of Section 6.2

What is the dimension of $W^\perp$ if $W$ is a line in $\mathbb{R}^{10}$?

What is $W^\perp$ if $W$ is the line $y = mx$ in $\mathbb{R}^{2}$?

If $W$ is the $x$-axis in $\mathbb{R}^{2}$, and $v = (7, -3)$, write $v$ as $v_{W} + v_{W}^\perp$. 
Summary of Section 6.2

• $W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w \text{ for all } w \in W \}$

• Facts:
  1. $W^\perp$ is a subspace of $\mathbb{R}^n$
  2. $(W^\perp)^\perp = W$
  3. $\dim W + \dim W^\perp = n$
  4. If $W = \text{Span}\{w_1, \ldots, w_k\}$ then $W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w_i \text{ for all } i \}$
  5. The intersection of $W$ and $W^\perp$ is $\{0\}$.

• To find $W^\perp$, find a basis for $W$, make those vectors the rows of a matrix, and find the null space.

• Every vector $v$ can be written uniquely as $v = v_W + v_W^\perp$ with $v_W$ in $W$ and $v_W^\perp$ in $W^\perp$