

# Announcements April 20

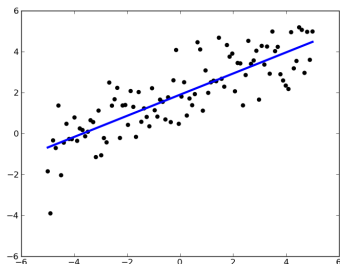
- Final exam originally scheduled for Tue Apr 28. Details TBA.
- Chapter 6 WeBWorkS are not graded.
- My office hours Monday 3-4, Wed 2-3, and by appointment
- Stay tuned for review sessions...
- TA office hours on Blue Jeans (you can go to any of these!)
  - ▶ Isabella Wed 11-12
  - ▶ Kyle Wed 3-5, Thu 1-3
  - ▶ Kalen Mon/Wed 1-2
  - ▶ Sidhanth Tue 10-12
- Supplemental problems & practice exams on master web site
- Counseling Center: <http://counseling.gatech.edu> ▶ Click

# Section 6.5

## Least Squares Problems

# Least Squares problems

What if we can't solve  $Ax = b$ ? How can we solve it as closely as possible?



To solve  $Ax = b$  as closely as possible, we orthogonally project  $b$  onto  $\text{Col}(A)$ ; call the result  $\hat{b}$ . Then solve  $Ax = \hat{b}$ . This is the *least squares solution* to  $Ax = b$ .

## Outline of Section 6.5

- The method of least squares
- Application to best fit lines/planes
- Application to best fit curves

## Least squares solutions

$A = m \times n$  matrix.

A **least squares solution** to  $Ax = b$  is an  $\hat{x}$  in  $\mathbb{R}^n$  so that  $A\hat{x}$  is as close as possible to  $b$ .

The error is  $\|A\hat{x} - b\|$ .

▶ Demo

## Least squares solutions

A **least squares solution** to  $Ax = b$  is an  $\hat{x}$  in  $\mathbb{R}^n$  so that  $A\hat{x}$  is as close as possible to  $b$ .

The error is  $\|A\hat{x} - b\|$ .

**Theorem.** The least squares solutions to  $Ax = b$  are the solutions to

$$(A^T A)x = (A^T b)$$

So this is just like what we did before when we were finding the projection of  $b$  onto  $\text{Col}(A)$ . But now we just solve and don't (necessarily) multiply the solution by  $A$ .

# Least squares solutions

## Example

**Theorem.** The least squares solutions to  $Ax = b$  are the solutions to

$$(A^T A)x = (A^T b)$$

Find the least squares solutions to  $Ax = b$  for this  $A$  and  $b$ :

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

What is the error?

# Least squares solutions

## Example

Formula:  $(A^T A)x = (A^T b)$

Find the least squares solution/error to  $Ax = b$ :

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$



## Least squares solutions

**Theorem.** Let  $A$  be an  $m \times n$  matrix. The following are equivalent:

1.  $Ax = b$  has a unique least squares solution for all  $b$  in  $\mathbb{R}^m$
2. The columns of  $A$  are linearly independent
3.  $A^T A$  is invertible

In this case the least squares solution is  $(A^T A)^{-1}(A^T b)$ .

# Application

## Best fit lines

**Problem.** Find the best-fit line through  $(0, 6)$ ,  $(1, 0)$ , and  $(2, 0)$ .

▶ Demo

# Best fit lines

## Poll

What does the best fit line minimize?

1. the sum of the squares of the distances from the data points to the line
2. the sum of the squares of the vertical distances from the data points to the line
3. the sum of the squares of the horizontal distances from the data points to the line
4. the maximal distance from the data points to the line

# Least Squares Problems

## More applications

Determine the least squares problem  $Ax = b$  to find the best parabola  $y = Cx^2 + Dx + E$  for the points:

$$(0, 0), (2, 0), (3, 0), (0, 1)$$

▶ Demo

# Least Squares Problems

## More applications

Determine the least squares problem  $Ax = b$  to find the best fit ellipse  $Cx^2 + Dxy + Ey^2 + Fx + Gy + H = 0$  for the points:

$$(0, 0), (2, 0), (3, 0), (0, 1)$$

Gauss invented the method of least squares to predict the orbit of the asteroid Ceres as it passed behind the sun in 1801.

# Least Squares Problems

## Best fit plane

Determine the least squares problem  $Ax = b$  to find the best fit linear function  $f(x, y) = Cx + Dy + E$

$x$	$y$	$f(x, y)$
1	0	0
0	1	1
-1	0	3
0	-1	4

## Summary of Section 6.5

- A **least squares solution** to  $Ax = b$  is an  $\hat{x}$  in  $\mathbb{R}^n$  so that  $A\hat{x}$  is as close as possible to  $b$ .
- The error is  $\|A\hat{x} - b\|$ .
- The least squares solutions to  $Ax = b$  are the solutions to  $(A^T A)x = (A^T b)$ .
- To find a best fit line/parabola/etc. write the general form of the line/parabola/etc. with unknown coefficients and plug in the given points to get a system of linear equations in the unknown coefficients.