Announcements April 1

- Class participation (Piazza polls) is optional for the rest of the semester.
- We will use Blue Jeans Meetings for the rest of the semester.
- The new schedule is on the web page.
- Midterm 3 on April 17
- WeBWorK 5.1 due tomorrow: Thu April 2.
- Official quiz on Friday on Canvas on 4.1, 4.2, 4.3, 5.1.
  It will be open all day Friday, but there will be a time limit.
- My office hours ~~Monday 3-4 and Wed 2-3 on Blue Jeans~~ by appt
  send email
- TA office hours on Blue Jeans (you can go to any of these!)
  - Isabella Mon 11-12, Wed 11-12
  - Kyle Wed 3-5, Thu 1-3
  - Kalen Mon/Wed 1-2
  - Sidhanth Tue 10-12
- Supplemental problems and practice exams on the master web site
- Counseling Center: http://counseling.gatech.edu Click

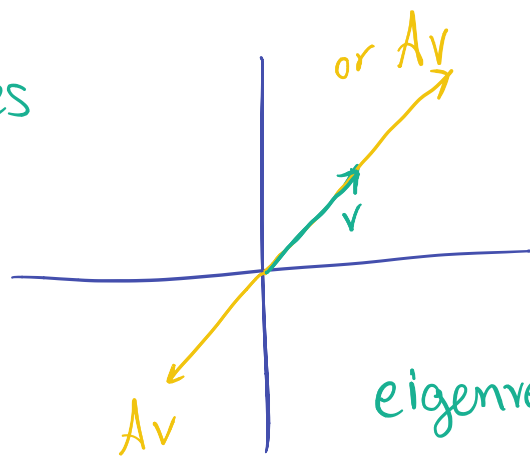
Section 5.2

The characteristic polynomial

Today

3) Find eigenvalues

A



eigenvector.

So far

1) Can check if $\lambda = 3$ is an eigenvalue.

2) Can find the 3-eigenspace

Characteristic polynomial

$Av = \lambda v$
$Av - \lambda I v = 0$
$(A - \lambda I)v = 0$

Recall:

$$\lambda \text{ is an eigenvalue of } A \iff \boxed{A - \lambda I} \text{ is not invertible}$$

So to find eigenvalues of A we solve

$$\boxed{\det(A - \lambda I) = 0}$$

The left hand side is a polynomial, the characteristic polynomial of A.

The roots of the characteristic polynomial are the eigenvalues of A.

The eigenrecipe

Say you are given an square matrix A.

Step 1. Find the eigenvalues of A by solving

$$\boxed{\det(A - \lambda I) = 0}$$

Today
Sec 5.2

Step 2. For each eigenvalue λ_i the λ_i-eigenspace is the solution to

$$(A - \lambda_i I)x = 0$$

Last class
Sec 5.1

To find a basis, find the vector parametric solution, as usual.

Characteristic polynomial

Find the characteristic polynomial and eigenvalues of

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (5-\lambda)(1-\lambda) - 4$$

$$= \lambda^2 - 6\lambda + 5 - 4$$

char poly

$$= \lambda^2 - 6\lambda + 1$$

$$\rightsquigarrow \boxed{\lambda = \frac{6 \pm \sqrt{36-4}}{2}}$$

$$= 3 \pm \sqrt{32}/2 \ldots$$

$$= 3 \pm \sqrt{8}$$

Characteristic polynomials, trace, and determinant

The trace of a matrix is the sum of the diagonal entries.

The characteristic polynomial of an $n \times n$ matrix A is a polynomial with leading term $(-1)^n$, next term $(-1)^{n-1}\mathrm{trace}(A)$, and constant term $\det(A)$:

$$(-1)^n \lambda^n + (-1)^{n-1} \underline{\mathrm{trace}(A)} \lambda^{n-1} + \cdots + \det(A)$$

So for a 2×2 matrix:

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A)$$

$$\begin{pmatrix} \boxed{5} & 2 \\ 2 & \boxed{1} \end{pmatrix} \rightsquigarrow \lambda^2 - \underline{6}\lambda + 1$$

trace $= 5 + 1$

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$A = \begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ -3 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & \boxed{+} & - \\ + & - & + \end{pmatrix}$$

What are the eigenvalues? Hint: Don't multiply everything out!

$$\det\begin{pmatrix} 7-\lambda & 0 & 3 \\ -3 & 2-\lambda & -3 \\ -3 & 0 & -1-\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} 7-\lambda & 3 \\ -3 & -1-\lambda \end{pmatrix}$$

$$= (2-\lambda)\big((7-\lambda)(-1-\lambda)+9\big)$$

$$= (2-\lambda)(\lambda^2 - 6\lambda + 2)$$

$$\boxed{\lambda = 2} \qquad \lambda = \frac{6 \pm \sqrt{36-8}}{2}$$

trace formula

$$-\lambda^3 + 8\lambda^2 + ? \;\; + \det$$

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ 4 & 2 & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} 7-\lambda & 0 & 3 \\ -3 & 2-\lambda & -3 \\ 4 & 2 & -\lambda \end{pmatrix}$$

$$+(7-\lambda)\det\begin{pmatrix} 2-\lambda & -3 \\ 2 & -\lambda \end{pmatrix} + 3\det\begin{pmatrix} -3 & 2-\lambda \\ 4 & 2 \end{pmatrix}$$

Answer: $\boxed{-\lambda^3 + 9\lambda^2 - 8\lambda}$

What are the eigenvalues? $\lambda = 0$ b/c:

$$-\lambda(\lambda^2 - 9\lambda + 8) \xrightarrow{\text{factor}} \lambda = 8, 1$$

$\lambda = 0$

$$\frac{9 \pm \sqrt{81-32}}{2}$$

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Answer:

$$-\lambda^3 + 3\lambda + 2$$

What are the eigenvalues? $\lambda = 2$

Hint: We already know one eigenvalue! Polynomial long division $\rightsquigarrow$

$$(\lambda - 2)(-\lambda^2 - 2\lambda - 1)$$

Don't really need long division: the first and last terms of the quadratic are easy to find; can guess and check the other term.

Without the hint, could use the rational root theorem: any integer root of a polynomial with leading coefficient ± 1 divides the constant term.

Eigenvalues

Triangular matrices

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \lambda = 1, 4, 0$$

$$\det \begin{pmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{pmatrix} = (1-\lambda)(4-\lambda)(6-\lambda)$$

$\lambda = 4$ (multiplicity 3)

$$\begin{pmatrix} 4 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}$$

$\rightsquigarrow \lambda = 1, 4, 6$

Algebraic multiplicity

The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example. Find the algebraic multiplicities of the eigenvalues for

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

example

I_n identity

$\lambda = 1$ has multipl. $= n$

char poly: $(1-\lambda)^n$

$\lambda = 0$ has mult 2

$\lambda = \pm 1$ have mult. 1.

$(\lambda - 1)(\lambda - 0)(\lambda + 1)(\lambda - 0)$

$(\lambda - 1)(\lambda - 0)^2(\lambda + 1)$

Fact. The sum of the algebraic multiplicities of the (real) eigenvalues of an $n \times n$ matrix is at most n.

$\boxed{\text{Later: dim eigenspace} \leq \text{alg mult}}$

Review of Section 5.2

True or false: every $n \times n$ matrix has an eigenvalue.

True or false: every $n \times n$ matrix has n distinct eigenvalues.

True or false: the nullity of $A - \lambda I$ is the dimension of the λ-eigenspace.

What are the eigenvalues for the standard matrix for a reflection?

Summary of Section 5.2

- The characteristic polynomial of A is $\det(A - \lambda I)$
- The roots of the characteristic polynomial for A are the eigenvalues
- Techniques for 3×3 matrices:
  - Don't multiply out if there is a common factor
  - If there is no constant term then factor out λ
  - If the matrix is triangular, the eigenvalues are the diagonal entries
  - Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - Use the geometry to determine an eigenvalue
- Given an square matrix A:
  - The eigenvalues are the solutions to $\det(A - \lambda I) = 0$
  - Each λ_i-eigenspace is the solution to $(A - \lambda_i I)x = 0$

Section 5.4

Diagonalization

Section 5.4 Outline

- Diagonalization
- Using diagonalization to take powers
- Algebraic versus geometric dimension

We understand diagonal matrices

We completely understand what diagonal matrices do to $\mathbb{R}^n$. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

stretches by 2 in x-dir

by 3 in y-dir

We have seen that it is useful to take powers of matrices: for instance in computing rabbit populations.

If A is diagonal, powers of A are easy to compute. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{10} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{pmatrix}$$

Powers of matrices that are similar to diagonal ones

What if A is not diagonal? Suppose want to understand the matrix

$$A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$$

geometrically? Or take it's 10th power? What would we do?

What if I give you the following equality:

eigenvalues

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$A \quad = \quad C \quad D \quad C^{-1}$$

This is called diagonalization.

eigenvectors

$$C \begin{pmatrix} 2^{100} & 0 \\ 0 & (1/2)^{100} \end{pmatrix} C^{-1}$$

$$\|$$

How does this help us understand A? Or find A^{10}?

$$A^2 = (CD\cancel{C^{-1}})(\cancel{C}DC^{-1}) = CD^2C^{-1} \qquad A^{100} = CD^{100}C^{-1}$$

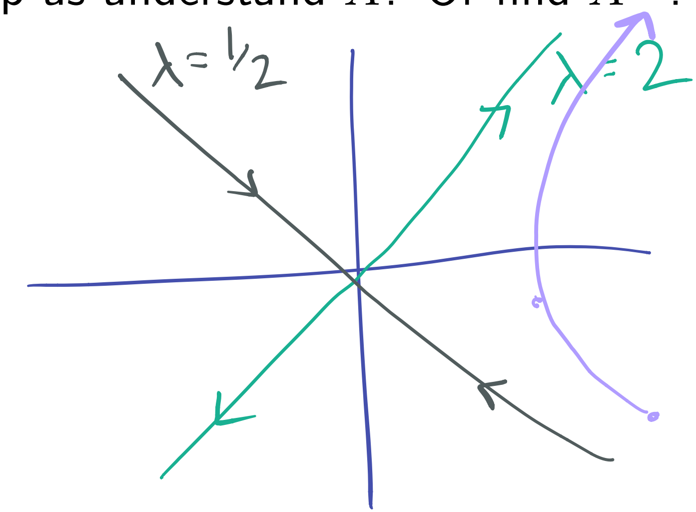
Powers of matrices that are similar to diagonal ones

What if I give you the following equality:

$$\underbrace{\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}}_{A} \underset{=}{=} \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}}_{C^{-1}}$$

This is called diagonalization.

How does this help us understand A? Or find A^{10}? Demo



Diagonalization

Suppose A is $n \times n$. We say that A is diagonalizable if we can write:

$$\boxed{A = CDC^{-1}} \qquad D = \text{diagonal}$$

We say that A is similar to D.

How does this factorization of A help describe what A does to $\mathbb{R}^n$? How does this help us take powers of A?

Understanding the rabbit example: since 2 is the largest eigenvalue, (almost) all other vectors get pulled towards that eigenvector. Compare with the example from the last slide.

Diagonalization

The recipe

Theorem. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

In this case

$$A = \underbrace{\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}}_{C^{-1}}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues (in order).

Why?

Example

Diagonalize if possible.

$\lambda=2$ eigenv $\quad$ $\lambda=-1$ eigenv.

$$\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$$

eigenvalues: 2, -1

$$\underbrace{\begin{pmatrix} 1 & ? \\ 0 & ? \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & ? \\ 0 & ? \end{pmatrix}^{-1}}_{C^{-1}}$$

eigenvectors $\lambda=2 \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Example

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Example

Diagonalize if possible.

$$\begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$$

Demo

More Examples

Diagonalize if possible.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Poll

Poll

Which are diagonalizable?

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Distinct Eigenvalues

Fact. If A has n distinct eigenvalues, then A is diagonalizable.

Why?

Non-Distinct Eigenvalues

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- $a_i =$ algebraic multiplicity of λ_i
- $d_i =$ dimension of λ_i eigenspace ("geometric multiplicity")

Then

1. $d_i \leq a_i$ for all i
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n$
   $\Leftrightarrow \Sigma a_i = n$ and $d_i = a_i$ for all i

So: if you find one eigenvalue where the geometric multiplicity is less than the algebraic multiplicity, the matrix is not diagonalizable.

Review of Section 5.4

True or false: If A is a 3×3 matrix with eigenvalues 0, 1, and 2, then A is diagonalizable.

True or false: It is possible for an eigenspace to be 0-dimensional.

Summary of Section 5.4

- A is diagonalizable if $A = CDC^{-1}$ where D is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $\Leftrightarrow$ the sum of the geometric dimensions of the eigenspaces in n
- If A has n distinct eigenvalues it is diagonalizable