Announcements April 13

- Midterm 3 on Friday
- WeBWorK 5.5 & 5.6 due Thu Apr 16.
- Survey about on-line learning on Canvas...
- My office hours Monday 3-4, Wed 2-3, and by appointment
- TA office hours on Blue Jeans (you can go to any of these!)
  - Isabella Wed 11-12
  - Kyle Wed 3-5, Thu 1-3
  - Kalen Mon/Wed 1-2
  - Sidhanth Tue 10-12
- Supplemental problems & practice exams on master web site
- Counseling Center: http://counseling.gatech.edu
Chapter 6
Orthogonality
Section 6.1

Dot products and Orthogonality
Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can’t solve $Ax = b$? How can we solve it as closely as possible?

The answer relies on orthogonality.
Outline

- Dot products
- Length and distance
- Orthogonality
Dot product

Say \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_n) \) are vectors in \( \mathbb{R}^n \)

\[
\begin{align*}
    u \cdot v &= \sum_{i=1}^{n} u_i v_i \\
    &= u_1 v_1 + \cdots + u_n v_n \\
    &= u^T v
\end{align*}
\]

Example. Find \((1, 2, 3) \cdot (4, 5, 6)\).

\[
\begin{align*}
    &= 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\
    &= 4 + 10 + 18 \\
    &= 32
\end{align*}
\]
Dot product

Some properties of the dot product

- $u \cdot v = v \cdot u$
- $(u + v) \cdot w = u \cdot w + v \cdot w$
- $(cu) \cdot v = c(u \cdot v)$
- $u \cdot u \geq 0$
- $u \cdot u = 0 \iff u = 0$

$$\begin{pmatrix} -1, -2, -3 \end{pmatrix} \cdot \begin{pmatrix} -1, -2, -3 \end{pmatrix}$$

$$= (-1)^2 + (-2)^2 + (-3)^2 > 0$$
Length

Let $v$ be a vector in $\mathbb{R}^n$

$$\|v\| = \sqrt{v \cdot v}$$

= length of $v$

Why? Pythagorean Theorem

Fact. $\|cv\| = |c| \cdot \|v\|$

$v$ is a unit vector of $\|v\| = 1$

Problem. Find the unit vector in the direction of $(1, 2, 3, 4)$.
Distance

The distance between \( v \) and \( w \) is the length of \( v - w \) (or \( w - v \!).

Problem. Find the distance between \((1,1,1)\) and \((1,4,-3)\).

\[
\begin{align*}
v - w &= (0, -3, 4) \\
\|v - w\| &= 5 = \sqrt{0^2 + (-3)^2 + 4^2}
\end{align*}
\]
Orthogonality

Fact. $u \perp v \iff u \cdot v = 0$

Why? Pythagorean theorem again!

\[ u \perp v \iff \|u\|^2 + \|v\|^2 = \|u - v\|^2 \]
\[ \iff u \cdot u + v \cdot v = u \cdot u - 2u \cdot v + v \cdot v \]
\[ \iff u \cdot v = 0 \]

Problem. Find a vector in $\mathbb{R}^3$ orthogonal to $(1, 2, 3)$.

$$\begin{align*}
(1, 2, 3) \cdot (1, 1, -1) & = 1 \cdot 1 + 2 \cdot 1 + 3 \cdot -1 \\
& = 0.
\end{align*}$$
Section 6.2
Orthogonal complements
Outline of Section 6.2

• Orthogonal complements
• Computing orthogonal complements
Orthogonal complements

$\mathbf{W} = \text{subspace of } \mathbb{R}^n$

$\mathbf{W}^\perp = \{ v \in \mathbb{R}^n \mid v \perp w \text{ for all } w \in \mathbf{W} \}$

**Question.** What is the orthogonal complement of a line in $\mathbb{R}^3$?

**Facts.**

1. $\mathbf{W}^\perp$ is a subspace of $\mathbb{R}^n$
2. $(\mathbf{W}^\perp)^\perp = \mathbf{W}$
3. $\dim \mathbf{W} + \dim \mathbf{W}^\perp = n$
4. If $\mathbf{W} = \text{Span}\{w_1, \ldots, w_k\}$ then $\mathbf{W}^\perp = \{ v \in \mathbb{R}^n \mid v \perp w_i \text{ for all } i \}$
5. The intersection of $\mathbf{W}$ and $\mathbf{W}^\perp$ is $\{0\}$. 
Orthogonal complements
Finding them

Problem. Let \( W = \text{Span}\{(1, 1, -1)\} \). Find the equation of the plane \( W^\perp \).

\[
W^\perp = \text{Nul}(1, 1, -1)
\]

Why?
\( V \) in \( \text{Nul}(1, 1, -1) \)
means \((1, 1, -1)(v_1, v_2, v_3) = 0\)

means \((1, 1, -1) \cdot v = 0\)

Equation of the plane
\[
(1, 1, -1)(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = 0
\]

\( x + y - z = 0 \)

Vect param form gives a basis
for \( W^\perp \)

\[
\{(1, 1, 0), (0, -1, 1)\}
\]
Orthogonal complements
Finding them

Problem. Let $W = \text{Span}\{(1, 1, -1), (-1, 2, 1)\}$. Find a system of equations describing the line $W^\perp$. Also find a basis for $W^\perp$.

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$$

Sys of eqns

\[ x + y - z = 0 \]
\[-x + 2y + z = 0 \]

Basis

Vect. param. form

\[ \rightarrow 1 \text{ vector} \]

orth. compl. of plane in $\mathbb{R}^3$ is a line.
Orthogonal complements

Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for $W$, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \iff x$ is orthogonal to each row of $A$
Orthogonal decomposition

**Fact.** Say $W$ is a subspace of $\mathbb{R}^n$. Then any vector $v$ in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where $v_W$ is in $W$ and $v_{W^\perp}$ is in $W^\perp$.

**Why?** Say that $w_1 + w_1' = w_2 + w_2'$ where $w_1$ and $w_2$ are in $W$ and $w_1'$ and $w_2'$ are in $W^\perp$. Then $w_1 - w_2 = w_2' - w_1'$. But the former is in $W$ and the latter is in $W^\perp$, so they must both be equal to 0.

Next time: Find $v_W$ and $v_{W^\perp}$. 
Orthogonal Projections

Many applications, including:
Review of Section 6.2

What is the dimension of $W^\perp$ if $W$ is a line in $\mathbb{R}^{10}$?

What is $W^\perp$ if $W$ is the line $y = mx$ in $\mathbb{R}^2$?

If $W$ is the $x$-axis in $\mathbb{R}^2$, and $v = (7, -3)$, write $v$ as $v_W + v_{W^\perp}$. 