Announcements April 13

- Midterm 3 on Friday

  Ch. 4 & 5
- WeBWorK 5.5 & 5.6 due Thu Apr 16.
- Survey about on-line learning on Canvas...
- My office hours Monday 3-4, Wed 2-3, and by appointment
- TA office hours on Blue Jeans (you can go to any of these!)
  - Isabella Wed 11-12
  - Kyle Wed 3-5, Thu 1-3
  - Kalen Mon/Wed 1-2
  - Sidhanth Tue 10-12
- Supplemental problems & practice exams on master web site
- Counseling Center: http://counseling.gatech.edu Click

Chapter 6

Orthogonality

Section 6.1

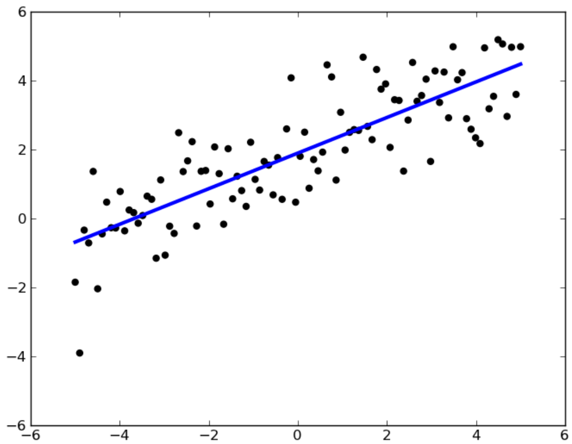
Dot products and Orthogonality

Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



The answer relies on orthogonality.

Outline

- Dot products
- Length and distance
- Orthogonality

Dot product

Say $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$

$$
\begin{aligned}
u \cdot v &= \sum_{i=1}^{n} u_i v_i \\
&= u_1 v_1 + \cdots + u_n v_n \\
&= u^T v
\end{aligned}
$$

Example. Find $(1, 2, 3) \cdot (4, 5, 6)$.

$$= 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 32 \end{pmatrix}$$

Dot product

Some properties of the dot product

- $u \cdot v = v \cdot u$
- $(u + v) \cdot w = u \cdot w + v \cdot w$
- $(cu) \cdot v = c(u \cdot v)$
- $u \cdot u \geq 0$
- $u \cdot u = 0 \Leftrightarrow u = 0$

$(-1, -2, -3) \cdot (-1, -2, -3)$

$= (-1)^2 + (-2)^2 + (-3)^2 \geq 0$

Length

Let v be a vector in $\mathbb{R}^n$

$$\|v\| = \sqrt{v \cdot v}$$
$$= \text{length of } v$$

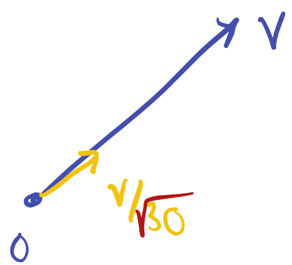
$\|(3,4)\|$
$= \sqrt{(3,4)\cdot(3,4)}$
$= \sqrt{3^2+4^2}$
$= \sqrt{25} = 5$

Why? Pythagorean Theorem

Fact. $\|cv\| = |c| \cdot \|v\|$

v is a unit vector of $\|v\| = 1$

Problem. Find the unit vector in the direction of $(1, 2, 3, 4)$.



$\|v\| = \sqrt{30}$

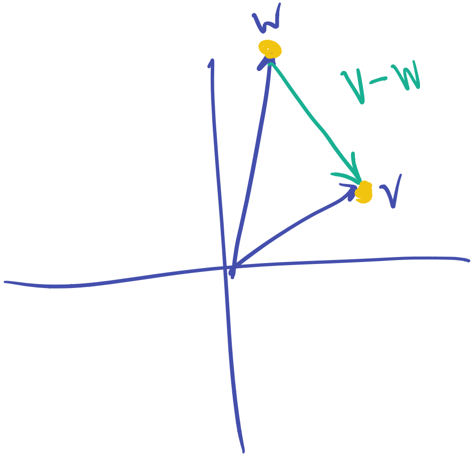
$v/\sqrt{30}$ is a unit vector

$(1/\sqrt{30}, 2/\sqrt{30}, 3/\sqrt{30}, 4/\sqrt{30})$

↳ Length $= 1^2+2^2+3^2+4^2$
$= \sqrt{1+4+9+16}$
$= \sqrt{30}$

Distance

The distance between v and w is the length of $v - w$ (or $w - v$!).



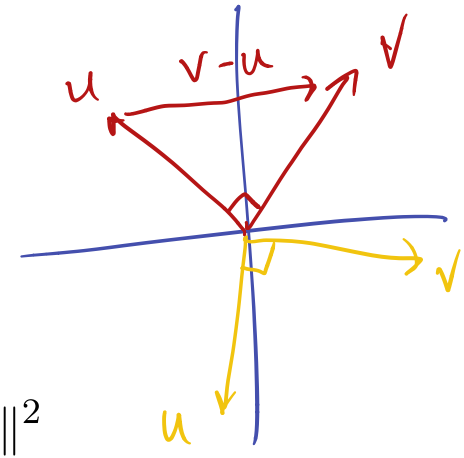
Problem. Find the distance between $\overset{v}{(1,1,1)}$ and $\overset{w}{(1,4,-3)}$.

$$v - w = (0, -3, 4)$$

$$\|v-w\| = 5 = \sqrt{0^2 + -3^2 + 4^2}$$

Orthogonality

Fact. $u \perp v \Leftrightarrow u \cdot v = 0$



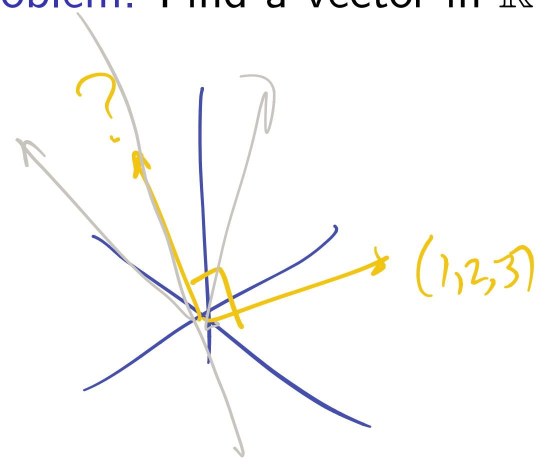
Why? Pythagorean theorem again!

$$\begin{aligned} u \perp v &\Leftrightarrow \|u\|^2 + \|v\|^2 = \|u - v\|^2 \\ &\Leftrightarrow \cancel{u \cdot u} + \cancel{v \cdot v} = \cancel{u \cdot u} - 2u \cdot v + \cancel{v \cdot v} \\ &\Leftrightarrow u \cdot v = 0 \end{aligned}$$

orthogonal to everything

$(0,0,0)$

Problem. Find a vector in $\mathbb{R}^3$ orthogonal to $(1, 2, 3)$.



$(1,2,3) \cdot \boxed{(1,1,-1)}$ or.... $(-1,1,1)$

$= 1 \cdot 1 + 2 \cdot 1 + 3 \cdot -1$

$= 0.$

Section 6.2

Orthogonal complements

Outline of Section 6.2

- Orthogonal complements
- Computing orthogonal complements

Orthogonal complements

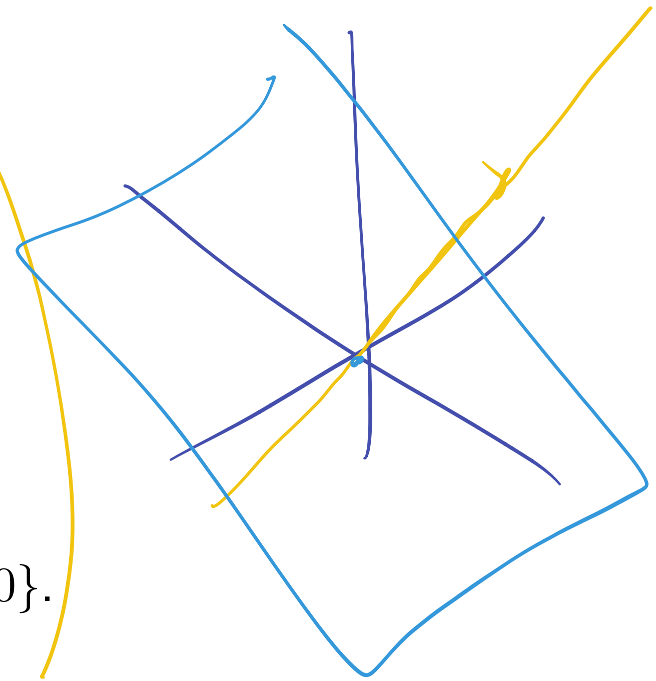
$W = \boxed{\text{subspace}}$ of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$?

Demo   Demo

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
   $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
5. The intersection of W and $W^\perp$ is $\{0\}$.



Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1,1,-1)\}$. Find the equation of the plane $W^\perp$.

$$W^\perp = \mathrm{Nul}\begin{pmatrix}1 & 1 & -1\end{pmatrix}$$

Equation of the plane

$$\begin{pmatrix}1 & 1 & -1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$$\boxed{x + y - z = 0}$$

Why?

v in $\mathrm{Nul}\begin{pmatrix}1 & 1 & -1\end{pmatrix}$

means $\begin{pmatrix}1 & 1 & -1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = 0$

means $\begin{pmatrix}1 & 1 & -1\end{pmatrix} \cdot v = 0$

Vect param form gives a basis for $W^\perp$

$$\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$$

Orthogonal complements

Finding them

plane

Problem. Let $W = \mathrm{Span}\{(1,1,-1),(-1,2,1)\}$. Find a system of equations describing the line $W^\perp$. Also find a basis for $W^\perp$.

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$$

Sys of eqns

$$x + y - z = 0$$
$$-x + 2y + z = 0$$

Basis

Vect. param. form

$\rightarrow$ 1 vector.

orth. compl. of plane in $\mathbb{R}^3$ is a line.

Orthogonal complements

Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

algebra $\Longleftrightarrow$ geometry

Orthogonal decomposition

Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

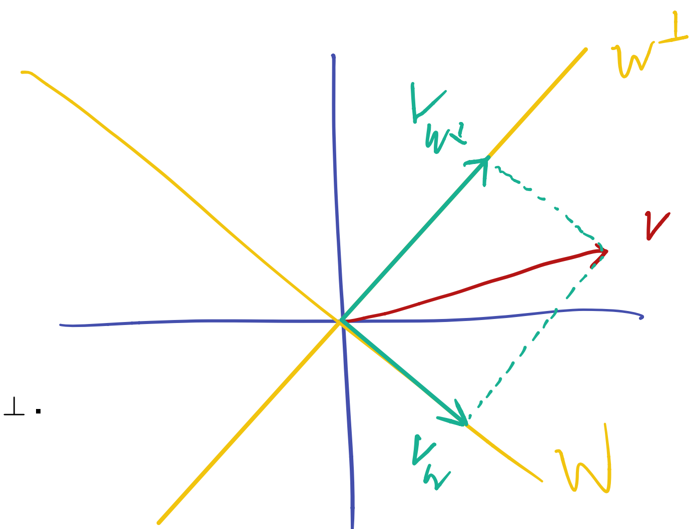
$$v = v_W + v_{W^\perp}$$

where v_W is in W and $v_{W^\perp}$ is in $W^\perp$.

~~**Why?** Say that $w_1 + w_1' = w_2 + w_2'$ where w_1 and w_2 are in W and w_1' and w_2' are in $W^\perp$. Then $w_1 - w_2 = w_2' - w_1'$. But the former is in W and the latter is in $W^\perp$, so they must both be equal to 0.~~

Demo

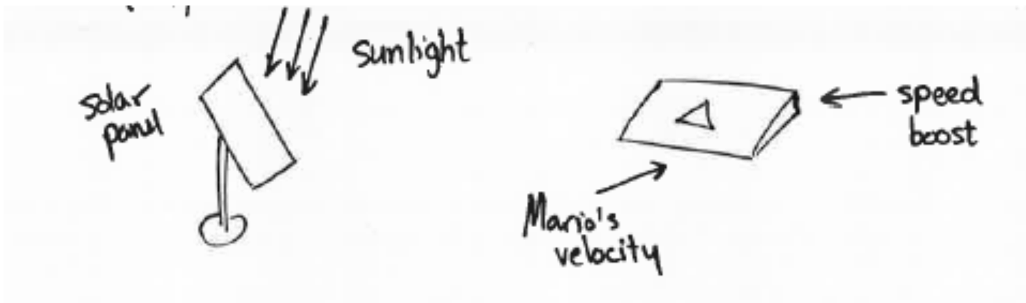
Demo



Next time: Find v_W and $v_{W^\perp}$.

Orthogonal Projections

Many applications, including:



Review of Section 6.2

What is the dimension of $W^\perp$ if W is a line in $\mathbb{R}^{10}$?

What is $W^\perp$ if W is the line $y = mx$ in $\mathbb{R}^2$?

If W is the x-axis in $\mathbb{R}^2$, and $v = (7, -3)$, write v as $v_W + v_{W^\perp}$.