Announcements April 15

• Midterm 3 on Friday

• WeBWorK 5.5 & 5.6 due Thu Apr 16.

• My office hours Monday 3-4, Wed 2-3, and by appointment

• See Canvas for review sessions...

• TA office hours on Blue Jeans (you can go to any of these!)
  ▶ Isabella Wed 11-12
  ▶ Kyle Wed 3-5, Thu 1-3
  ▶ Kalen Mon/Wed 1-2
  ▶ Sidhanth Tue 10-12

• Supplemental problems & practice exams on master web site

• Counseling Center: http://counseling.gatech.edu
Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can’t solve $Ax = b$? How can we solve it as closely as possible?

The answer relies on orthogonality.
Section 6.2
Orthogonal complements
Orthogonal complements

\[ W = \text{subspace of } \mathbb{R}^n \]
\[ W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w \text{ for all } w \in W \} \]

**Question.** What is the orthogonal complement of a line in \( \mathbb{R}^3 \)?

**Facts.**
1. \( W^\perp \) is a subspace of \( \mathbb{R}^n \)
2. \((W^\perp)^\perp = W\)
3. \(\dim W + \dim W^\perp = n\)
4. If \( W = \text{Span}\{w_1, \ldots, w_k\} \) then 
   \[ W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w_i \text{ for all } i \} \]
5. The intersection of \( W \) and \( W^\perp \) is \{0\}. 

Plane in \( \mathbb{R}^3 \) & vice versa
Orthogonal complements
Finding them

Problem. Let \( W = \text{Span}\{(1, 1, -1), (-1, 2, 1)\} \). Find a system of equations describing the line \( W^\perp \). And find basis.

\[ W^\perp = \text{Null} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix} \]

\[
\begin{pmatrix}
1 & 1 & -1 \\
-1 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 0
\]

means 1 \( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) in Null

2 \( (x, y, z) \perp \) rows
Orthogonal complements
Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for $W$, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \iff x$ is orthogonal to each row of $A$
Fact. Say $W$ is a subspace of $\mathbb{R}^n$. Then any vector $v$ in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where $v_W$ is in $W$ and $v_{W^\perp}$ is in $W^\perp$.

Why? Say that $w_1 + w'_1 = w_2 + w'_2$ where $w_1$ and $w_2$ are in $W$ and $w'_1$ and $w'_2$ are in $W^\perp$. Then $w_1 - w_2 = w'_2 - w'_1$. But the former is in $W$ and the latter is in $W^\perp$, so they must both be equal to 0.

Next time: Find $v_W$ and $v_{W^\perp}$. 
Section 6.3
Orthogonal projection
Outline of Section 6.3

- Orthogonal projections and distance
- A formula for projecting onto any subspace
- A special formula for projecting onto a line
- Matrices for projections
- Properties of projections
Orthogonal Projections

Let $v$ be a vector in $\mathbb{R}^n$ and $W$ a subspace of $\mathbb{R}^n$.

The **orthogonal projection** of $v$ onto $W$ the vector obtained by drawing a line segment from $v$ to $W$ that is perpendicular to $W$.

**Fact.** The following three things are all the same:

- The orthogonal projection of $v$ onto $W$
- The vector $v_W$ (the $W$-part of $v$)
- The closest vector in $W$ to $v$
Orthogonal Projections

Theorem. Let $W = \text{Col}(A)$. For any vector $v$ in $\mathbb{R}^n$, the equation

$$A^T Ax = A^T v$$

is consistent and the orthogonal projection $v_W$ is equal to $Ax$ where $x$ is any solution.

\[ \text{proj of } v \text{ to } W = v_W = A (\text{any soln to } A^T Ax = A^T v) \]

If $W = \text{Span}\{ (\frac{1}{2}, \frac{3}{4}) \}$, make $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

$A^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

or: rows of $A^T$ are...cols of $A$

or: cols of $A^T$ are rows of $A$

ij entry of $A^T$ is the...ji entry of $A$
Orthogonal Projections

**Theorem.** Let $W = \text{Col}(A)$. For any vector $v$ in $\mathbb{R}^n$, the equation
\[
A^T A x = A^T v
\]
is consistent and the orthogonal projection $v_W$ is equal to $A x$ where $x$ is any solution.

**Why?** Choose $\hat{x}$ so that $A \hat{x} = v_W$. We know $v - v_W = v - A \hat{x}$ is in $W^\perp = \text{Nul}(A^T)$ and so
\[
0 = A^T (v - A \hat{x}) = A^T v - A^T A \hat{x}
\]
\[\Rightarrow A^T A \hat{x} = A^T v\]
Orthogonal Projections

Theorem. Let \( W = \text{Col}(A) \). For any vector \( v \) in \( \mathbb{R}^n \), the equation

\[
A^T Ax = A^T v
\]

is consistent and the orthogonal projection \( v_W \) is equal to \( Ax \) where \( x \) is any solution.

What does the theorem give when \( W = \text{Span}\{u\} \) is a line?

\[
A = u
\]

\[
A^T A = u^T u = u \cdot u = ||u||^2
\]

\[
A^T v = u^T v = u \cdot v
\]

So we solve \( (u \cdot u)x = u \cdot v \) multiply by \( A \)

Solve: \( x = \frac{u \cdot v}{u \cdot u} \) Multiply by \( A \):
Orthogonal Projection onto a line

Special case. Let $L = \text{Span}\{u\}$. For any vector $v$ in $\mathbb{R}^n$ we have:

$$v_L = \frac{u \cdot v}{u \cdot u} u$$

Find $v_L$ and $v_{L\perp}$ if $v = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ and $u = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

$U \cdot V = U^TV$

$W = L = \text{Span}\{u\}$

$V_L = \frac{u \cdot v}{u \cdot u} u = \frac{-2}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -2/3 \end{pmatrix}$

$V_{L\perp} = \begin{pmatrix} -2 \\ -3 \\ -2 \end{pmatrix} - \begin{pmatrix} 2/3 \\ -2/3 \\ -2/3 \end{pmatrix} = \ldots$
Orthogonal Projections

Theorem. Let $W = \text{Col}(A)$. For any vector $v$ in $\mathbb{R}^n$, the equation

$$A^T Ax = A^T v$$

is consistent and the orthogonal projection $v_W$ is equal to $Ax$ where $x$ is any solution.

Example. Find $v_W$ if $v = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Steps. Find $A^T A$ and $A^T v$, then solve for $x$, then compute $Ax$.

Question. How far is $v$ from $W$?
Example. Find $v_W$ if $v = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

Steps. Find $A^T A$ and $A^T v$, then solve for $x$, then compute $Ax$.

$A^T A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$A^T b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$

Solve $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x = \begin{pmatrix} 10 \\ 11 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 11 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 2 \\ 11 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 2 \\ 11 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 0 & 1 & -3 \\ 2 & 0 & -12 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 11 \\ 2 & 0 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{pmatrix}$

$\rightarrow x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Question. How far is $v$ from $W$?

$\Vert v_W \Vert = \Vert v - v_W \Vert = \Vert \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} \Vert = \Vert \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \Vert = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$
Orthogonal Projections

**Theorem.** Let $W = \text{Col}(A)$. For any vector $v$ in $\mathbb{R}^n$, the equation

$$A^T Ax = A^T v$$

is consistent and the orthogonal projection $v_W$ is equal to $Ax$ where $x$ is any solution.

**Special case.** If the columns of $A$ are independent then $A^T A$ is invertible, and so

$$v_W = A(A^T A)^{-1} A^T v.$$  

Why? The $x$ we find tells us which linear combination of the columns of $A$ gives us $v_W$. If the columns of $A$ are independent, there’s only one linear combination.
Matrices for projections

Fact. If the columns of \( A \) are independent and \( W = \text{Col}(A) \) and \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) is orthogonal projection onto \( W \) then the standard matrix for \( T \) is:

\[
A(A^T A)^{-1} A^T.
\]

Why?

Example. Find the standard matrix for orthogonal projection of \( \mathbb{R}^3 \) onto \( W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \)
Summary of Section 6.3

- The orthogonal projection of $v$ onto $W$ is $v_W$.
- $v_W$ is the closest point in $W$ to $v$.
- The distance from $v$ to $W$ is $\|v_{W^\perp}\|$.
- **Theorem.** Let $W = \text{Col}(A)$. For any $v$, the equation $A^T Ax = A^T v$ is consistent and $v_W$ is equal to $Ax$ where $x$ is any solution.
- **Special case.** If $L = \text{Span}\{u\}$ then $v_L = \frac{u \cdot v}{u \cdot u} u$.
- **Special case.** If the columns of $A$ are independent then $A^T A$ is invertible, and so $v_W = A(A^T A)^{-1} A^T v$.
- When the columns of $A$ are independent, the standard matrix for orthogonal projection to $\text{Col}(A)$ is $A(A^T A)^{-1} A^T$.
- Let $W$ be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(v) = v_W$. Then
  - $T$ is a linear transformation
  - etc.
- If $P$ is the standard matrix then
  - The 1–eigenspace of $P$ is $W$ (unless $W = 0$)
  - etc.