Announcements April 15

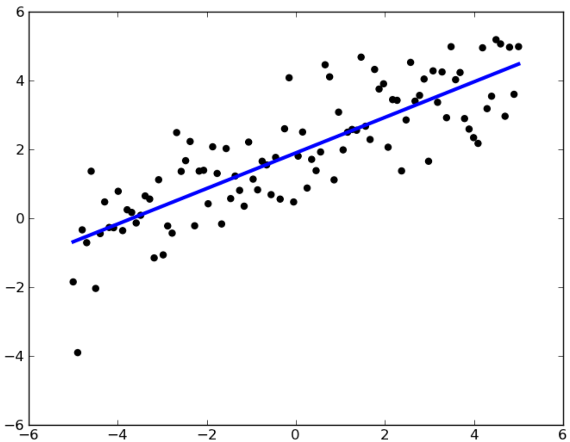
- Midterm 3 on Friday 75 mins, 24 hrs.
- WeBWorK 5.5 & 5.6 due Thu Apr 16.
- My office hours Monday 3-4, Wed 2-3, and by appointment
- See Canvas for review sessions... Kyle
- TA office hours on Blue Jeans (you can go to any of these!)
  - Isabella Wed 11-12
  - Kyle Wed 3-5, Thu 1-3
  - Kalen Mon/Wed 1-2
  - Sidhanth Tue 10-12
- Supplemental problems & practice exams on master web site
- Counseling Center: http://counseling.gatech.edu Click

Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



The answer relies on orthogonality.

Section 6.2

Orthogonal complements

Orthogonal complements

W = subspace of $\mathbb{R}^n$

$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$?

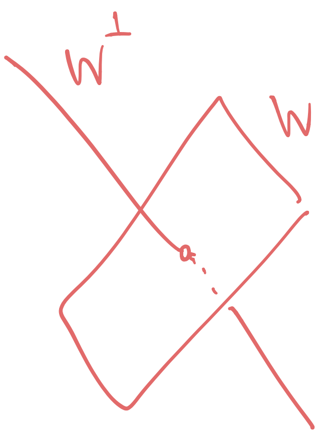
plane in $\mathbb{R}^3$

& vice versa

▸ Demo ▸ Demo

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
5. The intersection of W and $W^\perp$ is $\{0\}$.



Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1,1,-1),(-1,2,1)\}$. Find a system of equations describing the line $W^\perp$. And find basis.

$$W^\perp = \mathrm{Null}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ \hline -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

means ① $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in Null

② $(x,y,z) \perp$ rows

Orthogonal complements

Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

Orthogonal decomposition

Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where v_W is in W and $v_{W^\perp}$ is in $W^\perp$.

Find v_W by orth. proj.

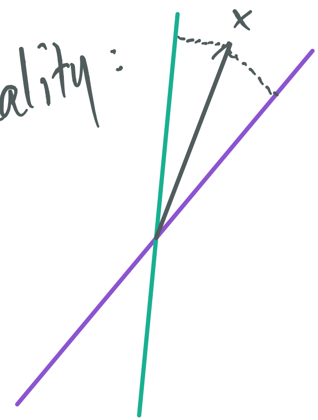
Then $v_{W^\perp} = v - v_W$

~~Why? Say that $w_1 + w_1' = w_2 + w_2'$ where w_1 and w_2 are in W and w_1' and w_2' are in $W^\perp$. Then $w_1 - w_2 = w_2' - w_1'$. But the former is in W and the latter is in $W^\perp$, so they must both be equal to 0.~~

Demo

Demo

need orthogonality:



Next time: Find v_W and $v_{W^\perp}$.

Section 6.3

Orthogonal projection

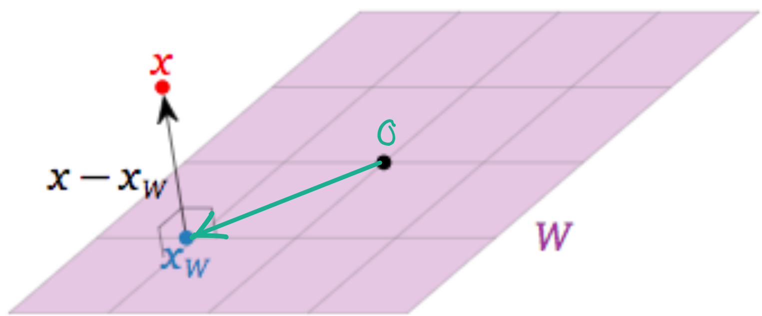
Outline of Section 6.3

- Orthogonal projections and distance
- A formula for projecting onto any subspace
- A special formula for projecting onto a line
- ~~Matrices for projections~~
- ~~Properties of projections~~

Orthogonal Projections

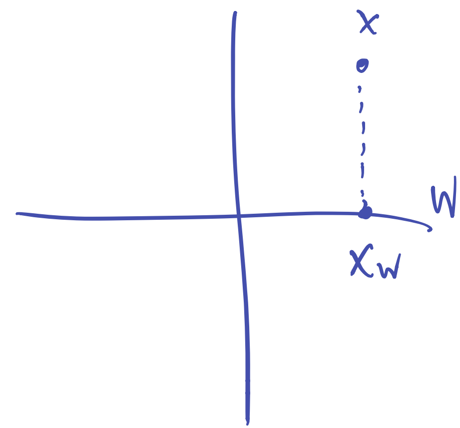
Let v be a vector in $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^n$.

The orthogonal projection of v onto W the vector obtained by drawing a line segment from v to W that is perpendicular to W.



Fact. The following three things are all the same:

- The orthogonal projection of v onto W
- The vector v_W (the W-part of v)
- The closest vector in W to v



Orthogonal Projections

If $W = \mathrm{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\}$

make $A = \begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}$

Theorem. Let $W = \mathrm{Col}(A)$. For any vector v in $\mathbb{R}^n$, the equation

$$A^T A x = A^T v$$

T = transpose

is consistent and the orthogonal projection v_W is equal to Ax where x is any solution.

proj. of v to W $= v_W = A(\text{any soln to } A^T A x = A^T v)$

$A^T = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ $\qquad A = \begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}$

or: rows of A^T are...cols of A

or: cols of A^T are rows of A

ij entry of A^T is the...ji entry of A

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector v in $\mathbb{R}^n$, the equation

$$A^T A x = A^T v$$

is consistent and the orthogonal projection v_W is equal to Ax where x is any solution.

Why? Choose $\widehat{x}$ so that $A\widehat{x} = v_W$. We know $\overset{v_{W^\perp}}{\overset{\|}{v - v_W}} = v - A\widehat{x}$ is in $W^\perp = \mathrm{Nul}(A^T)$ and so

$$0 = A^T(v - A\widehat{x}) = A^T v - A^T A\widehat{x}$$

$$\rightsquigarrow A^T A\widehat{x} = A^T v$$

Orthogonal Projections

$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Theorem. Let $W = \mathrm{Col}(A)$. For any vector v in $\mathbb{R}^n$, the equation

$$A^T A x = A^T v$$

is consistent and the orthogonal projection v_W is equal to Ax where x is any solution.

Span$\{u\}$ $=$ W

What does the theorem give when $W = \mathrm{Span}\{u\}$ is a line?

column vector

$A = u$

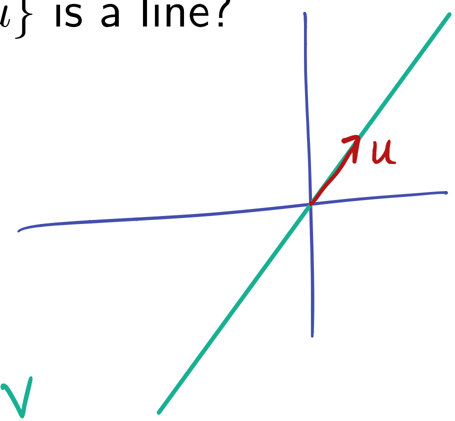
$A^T A = u^T u = u \cdot u = \|u\|^2$

$A^T v = u^T v = u \cdot v$

So we solve $(u \cdot u)x = u \cdot v$

multiply by A

Solve: $x = \dfrac{u \cdot v}{u \cdot u}$ Multiply by A: $\boxed{\dfrac{u \cdot v}{u \cdot u} u}$



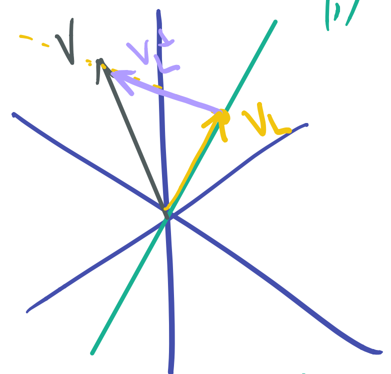
Orthogonal Projection onto a line

Special case. Let $L = \mathrm{Span}\{u\}$. For any vector v in $\mathbb{R}^n$ we have:

$$v_L = \frac{u \cdot v}{u \cdot u} u$$

$u \cdot v = u^T v$

$L = \mathrm{Span}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$



$W = L = \mathrm{span}\{u\}$

Find v_L and $v_{L^\perp}$ if $v = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ and $u = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

scalar   vector

$$v_L = \frac{u \cdot v}{u \cdot u} u = \frac{-2}{3}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -2/3 \\ -2/3 \end{pmatrix}$$

$$v_{L^\perp} = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} - \begin{pmatrix} 2/3 \\ -2/3 \\ -2/3 \end{pmatrix} = \ldots$$

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector v in $\mathbb{R}^n$, the equation

$$A^T A x = A^T v$$

is consistent and the orthogonal projection v_W is equal to Ax where x is any solution.

Example. Find v_W if $v = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Steps. Find $A^T A$ and $A^T v$, then solve for x, then compute Ax.

Question. How far is v from W?

Orthogonal Projections

$A^T A x = A^T v$ ← the vector you are projecting

↑ variable

Example. Find v_W if $v = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Steps. Find $A^T A$ and $A^T v$, then solve for x, then compute Ax.

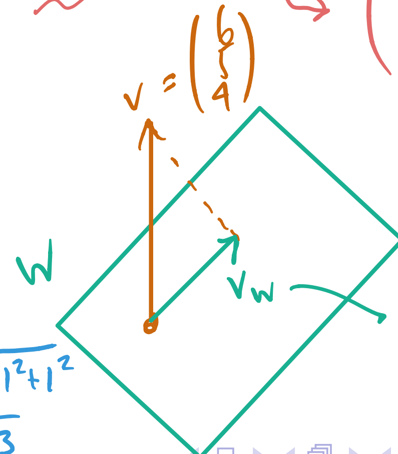
$$A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad A^T b = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$$

Solve $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$

$$\rightsquigarrow \left(\begin{array}{cc|c} 2 & 1 & 10 \\ 1 & 2 & 11 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 11 \\ 2 & 1 & 10 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 11 \\ 0 & -3 & -12 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 11 \\ 0 & 1 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right)$$

$$\rightsquigarrow x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$Ax = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}$$



$$\begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}$$

Question. How far is v from W?

$$\|v_{W^\perp}\| = \|v - v_W\|$$

$$= \left\| \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\| = \sqrt{(-1)^2 + 1^2 + 1^2}$$

$$= \sqrt{3}$$

Orthogonal Projections

Same Thm

Theorem. Let $W = \mathrm{Col}(A)$. For any vector v in $\mathbb{R}^n$, the equation

$$A^T A x = A^T v$$

is consistent and the orthogonal projection v_W is equal to Ax where x is any solution.

Special case. If the columns of A are independent then $A^T A$ is invertible, and so

$$v_W = A(A^T A)^{-1} A^T v.$$

Why? The x we find tells us which linear combination of the columns of A gives us v_W. If the columns of A are independent, there's only one linear combination.

Matrices for projections

Fact. If the columns of A are independent and $W = \mathrm{Col}(A)$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is orthogonal projection onto W then the standard matrix for T is:

$$A(A^T A)^{-1} A^T.$$

std matrix for orthog proj.

Why?

Example. Find the standard matrix for orthogonal projection of $\mathbb{R}^3$ onto $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Summary of Section 6.3

- The orthogonal projection of v onto W is v_W
- v_W is the closest point in W to v.
- The distance from v to W is $\|v_{W^\perp}\|$.
- Theorem. Let $W = \mathrm{Col}(A)$. For any v, the equation $A^T A x = A^T v$ is consistent and v_W is equal to Ax where x is any solution.
- Special case. If $L = \mathrm{Span}\{u\}$ then $v_L = \frac{u \cdot v}{u \cdot u} u$
- Special case. If the columns of A are independent then $A^T A$ is invertible, and so $v_W = A(A^T A)^{-1} A^T v$
- When the columns of A are independent, the standard matrix for orthogonal projection to $\mathrm{Col}(A)$ is $A(A^T A)^{-1} A^T$
- Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(v) = v_W$. Then
  - T is a linear transformation
  - etc.
- If P is the standard matrix then
  - The 1–eigenspace of P is W (unless $W = 0$)
  - etc.