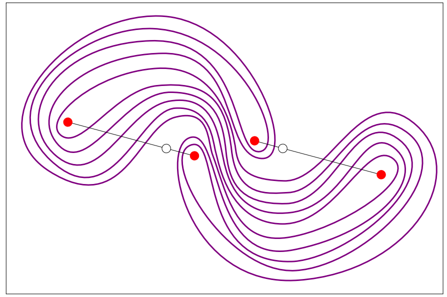
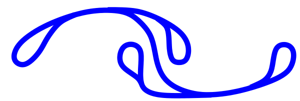
Announcements April 6

- Midterm 3 on April 17
- WeBWorK 5.5 & 5.6 due Thu Apr 9.
- Quiz 8am Fri - 8am Sat on 5.2 & 5.4.
  Expect less-Googlable questions.
- My office hours Monday 3-4, Wed 2-3, and by appointment
- TA office hours on Blue Jeans (you can go to any of these!)
  - Isabella Wed 11-12
  - Kyle Wed 3-5, Thu 1-3
  - Kalen Mon/Wed 1-2
  - Sidhanth Tue 10-12
- Supplemental problems and practice exams on the master web site
- Counseling Center: http://counseling.gatech.edu Click

Taffy pullers

How efficient is this taffy puller?

If you run the taffy puller, the taffy starts to look like the shape on the right. Every rotation of the machine changes the number of strands of taffy by a matrix:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

The largest eigenvalue λ of this matrix describes the efficiency of the taffy puller. With every rotation, the number of strands multiplies by λ.

Section 5.4

Diagonalization

We understand diagonal matrices

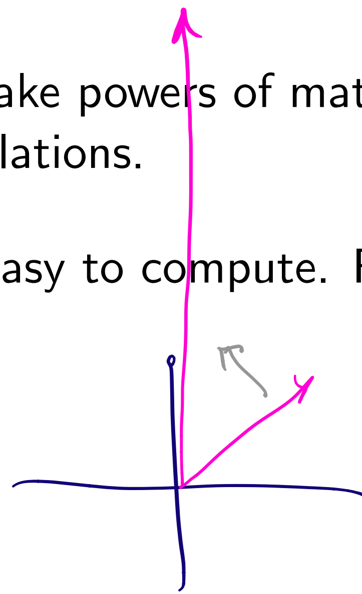
We completely understand what diagonal matrices do to $\mathbb{R}^n$. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

We have seen that it is useful to take powers of matrices: for instance in computing rabbit populations.

If A is diagonal, powers of A are easy to compute. For example:

$$\boxed{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{10} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{pmatrix}}$$



Powers of matrices that are similar to diagonal ones

What if A is not diagonal? Suppose want to understand the matrix

$$A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$$

geometrically? Or take it's 10th power? What would we do?

What if I give you the following equality:

$$\begin{array}{ccccccc} \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} & = & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \\ A & = & C & D & C^{-1} \end{array}$$

This is called diagonalization.

How does this help us understand A? Or find A^{10}?

Powers of matrices that are similar to diagonal ones

What if I give you the following equality: eigenvalues

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$A \quad = \quad C \quad D \quad C^{-1}$$

eigenvectors

This is called diagonalization.

How does this help us understand A? Or find A^{10}? Demo

$$A^3 = (CD\cancel{C^{-1}})(\cancel{C}D\cancel{C^{-1}})(\cancel{C}DC^{-1})$$

$$= C D^{\cancel{3}\,100} C^{-1} = C \begin{pmatrix} 2^{\cancel{3}\,100} & 0 \\ 0 & (1/2)^{\cancel{3}\,100} \end{pmatrix} C^{-1}$$

Diagonalization

The recipe

Theorem. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

choose the order but ...
make sure these match

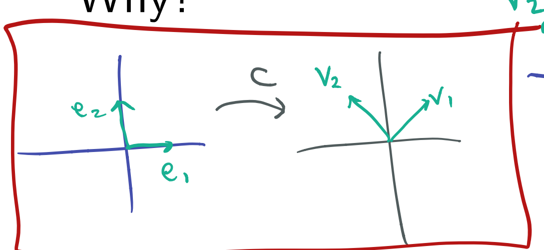
In this case

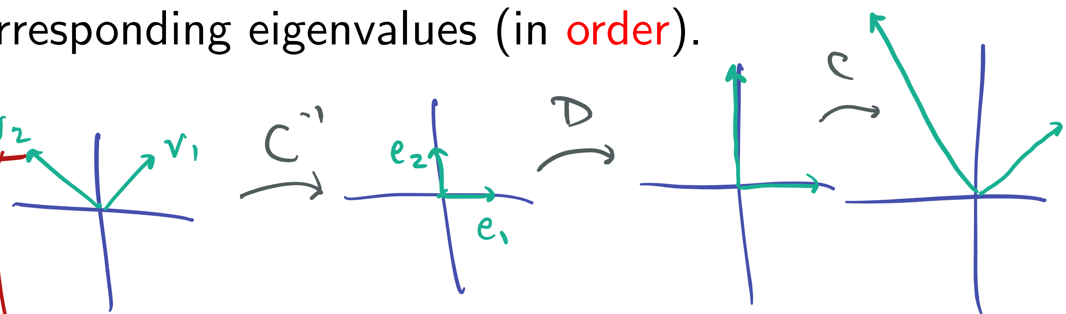
$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}$$

$$= \qquad C \qquad\qquad D \qquad\qquad C^{-1}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues (in order).

Why?





Example

Diagonalize if possible.

$$A = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$$

Triang. matrix $\rightsquigarrow$ $\lambda = 2, -1$

eigenvector

2-eigenspace: $A - 2I = \begin{pmatrix} 0 & 6 \\ 0 & -3 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} \boxed{0 \quad 1} \\ 0 \quad 0 \end{pmatrix}$ $\quad \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

-1-eigenspace: $A - (-I) = \begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}^{-1}$$

or $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^{-1}$

or $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}^{-1}$

Example

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Example. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ has one eigenval. but 2 indep eigenvecs $\rightsquigarrow$ diag'able. (already diagonal!)

Eigenvalues: $\lambda = 3$ (alg mult $= 2$)

3-Eigenspace: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow$ 1-dim eigenspace.

basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$\left(\quad \right) \left(\quad \right) \left(\quad \right)$

need two indep. eigenvectors

FAIL!

More Examples

Diagonalize if possible.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Same: 1 Find eigenvals

2 Find basis for each eigenspace

$$\begin{pmatrix} | & | & | \end{pmatrix}\begin{pmatrix} \square & & \\ & \square & \\ & & \square \end{pmatrix}\begin{pmatrix} | & | & | \end{pmatrix}^{-1}$$

Distinct Eigenvalues

Fact. If A has n distinct eigenvalues, then A is diagonalizable.

Why?

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^{-1}$$

Only thing can go wrong:
alg mult > 1.

3 has alg mult 2

0 has mult 1.

$$(\lambda-3)^2 \lambda$$

Non-Distinct Eigenvalues

$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ 3 has alg mult 2 but dim of 3 e-space is $\underline{1<2}$

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity")

Then

1. $d_i \le a_i$ for all i
2. A is diagonalizable $\Leftrightarrow \boxed{\Sigma d_i = n}$

   $\Leftrightarrow \Sigma a_i = n$ and $d_i = a_i$ for all i

So: if you find one eigenvalue where the ~~geometric multiplicity~~ dim of eigenspace is less than the algebraic multiplicity, the matrix is not diagonalizable.

$\boxed{(\lambda-3)(\lambda-5)^3(\lambda-7)^2}$ → 3 has alg mult 1
5 has alg mult 3
7 2

Section 5.5

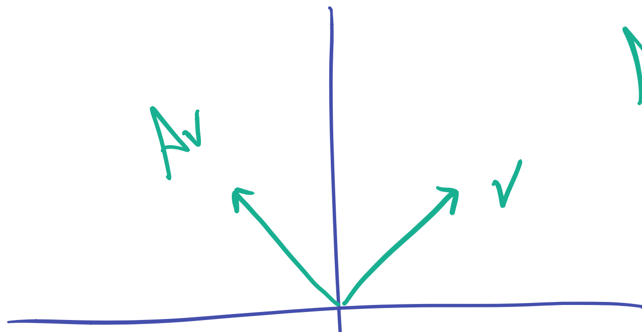
Complex Eigenvalues

Eigenvalues for rotations?

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

What are the eigenvectors and eigenvalues for rotation of $\mathbb{R}^2$ by $\pi/2$ (counterclockwise)?

No real eigenvectors!



No real eigenvalues!

Demo

eigenvals.

$$A=\begin{pmatrix}0&-1\\1&0\end{pmatrix} \qquad \det\begin{pmatrix}-\lambda&-1\\1&-\lambda\end{pmatrix}=\lambda^2+1$$

$$\frac{-0\pm\sqrt{0^2-4}}{2}=\pm\sqrt{-1}=\pm i$$

Imaginary numbers

Problem. When solving polynomial equations, we often run up against the issue that we can't take the square root of a negative number:

$$x^2 + 1 = 0$$

Solution. Take square roots of negative numbers:

$$x = \pm\sqrt{-1}$$

We usually write $\sqrt{-1}$ as i (for "imaginary"), so $x = \pm i$.

Now try solving these:

$$x^2 + 3 = 0$$

$$\sqrt{3 \cdot -1} = \sqrt{3}\sqrt{-1} = 3i$$

$$x^2 - x + 1 = 0$$

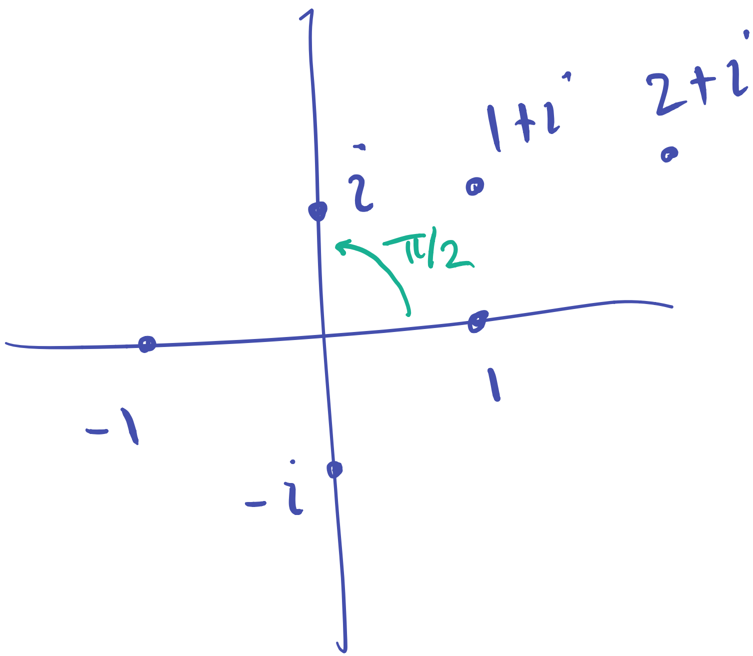
$$\frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}\, i}{2}$$

Complex numbers

The complex numbers are the numbers

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\} \longleftrightarrow \mathbb{R}^2$$

We can conjugate complex numbers: $\overline{a + bi} = a - bi$



$$\overline{7+3i} = 7-3i$$

$$\overline{7-3i} = 7+3i$$

Complex numbers and polynomials

Fundamental theorem of algebra. Every polynomial of degree n has exactly n complex roots (counted with alg. multiplicity).

real coeffs

Fact. If z is a root of a real polynomial then $\overline{z}$ is also a root.

So what are the possibilities for degree 2, 3 polynomials? ← next time

So if $7+3i$ is an eigenval of a real matrix, $7-3i$ is also

Complex eigenvalues

Say A is a square matrix with real entries.

We can now find complex eigenvectors and eigenvalues.

Fact. If λ is an eigenvalue of A with eigenvector v then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{v}$.

Why?

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

See above $\lambda = \pm i$

$i \cdot \begin{pmatrix} -1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ -1 \end{pmatrix}$

i-eigenspace: $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$

$\xrightarrow{\text{RREF}} \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 \\ +i \end{pmatrix}$ eigenvector

$-i$ eigenvector: $\begin{pmatrix} -1 \\ -i \end{pmatrix}$ conjugate!

Three shortcuts for complex eigenvectors

Suppose we have a 2×2 matrix with complex eigenvalue λ.

(1) We do not need to row reduce $A - \lambda I$ by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

the eigenvector is

$$A = \begin{pmatrix} -y \\ x \end{pmatrix}$$

(3) Also, we get the other eigenvalue/eigenvector pair for free: conjugation.

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$