Announcements April 6

- Midterm 3 on April 17
- WeBWorK 5.5 & 5.6 due Thu Apr 9.
- Quiz 8am Fri - 8am Sat on 5.2 & 5.4. Expect less-Googlable questions.
- My office hours Monday 3-4, Wed 2-3, and by appointment
- TA office hours on Blue Jeans (you can go to any of these!)
  - Isabella Wed 11-12
  - Kyle Wed 3-5, Thu 1-3
  - Kalen Mon/Wed 1-2
  - Sidhanth Tue 10-12
- Supplemental problems and practice exams on the master web site
- Counseling Center: http://counseling.gatech.edu
Taffy pullers

How efficient is this taffy puller?

If you run the taffy puller, the taffy starts to look like the shape on the right. Every rotation of the machine changes the number of strands of taffy by a matrix:

$$
\begin{pmatrix}
1 & 0 & 2 \\
2 & 1 & 2 \\
4 & 2 & 3 \\
\end{pmatrix}
$$

The largest eigenvalue $\lambda$ of this matrix describes the efficiency of the taffy puller. With every rotation, the number of strands multiplies by $\lambda$. 
Section 5.4
Diagonalization
We understand diagonal matrices

We completely understand what diagonal matrices do to $\mathbb{R}^n$. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

We have seen that it is useful to take powers of matrices: for instance in computing rabbit populations.

If $A$ is diagonal, powers of $A$ are easy to compute. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{10} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{pmatrix}$$
Powers of matrices that are similar to diagonal ones

What if $A$ is not diagonal? Suppose want to understand the matrix

$$A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$$

g eo metrically? Or take it’s 10th power? What would we do?

What if I give you the following equality:

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$A = CD C^{-1}$$

This is called diagonalization.

How does this help us understand $A$? Or find $A^{10}$?
Powers of matrices that are similar to diagonal ones

What if I give you the following equality:

\[
\begin{pmatrix}
\frac{5}{4} & \frac{3}{4} \\
\frac{3}{4} & \frac{5}{4}
\end{pmatrix} = \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
2 & 0 \\
0 & \frac{1}{2}
\end{pmatrix} \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}^{-1}
\]

\[A = CD C^{-1}\]

This is called diagonalization.

How does this help us understand \(A\)? Or find \(A^{10}\)?

\[A^3 = (CDC^{-1})(CDC^{-1})(CDC^{-1}) = CDC^{-1} = C \begin{pmatrix}
2 & 0 \\
0 & \frac{1}{2}
\end{pmatrix} C^{-1}\]
Diagonalization

The recipe

**Theorem.** A is diagonalizable $\iff$ $A$ has $n$ linearly independent eigenvectors.

In this case

$$A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_n & \\ & & & 1 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}^{-1} = \begin{pmatrix} C & D & C^{-1} \end{pmatrix}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues (in order).

Why?
Example

Diagonalize if possible.

\[ A = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix} \]

Triangular matrix \( \implies \lambda = 2, -1 \)

2-eigenspace: \( A - 2I = \begin{pmatrix} 0 & 6 \\ 0 & -3 \end{pmatrix} \implies \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \)

-1-eigenspace: \( A - (-1)I = \begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix} \implies \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \)

or \( \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \)

or \( \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \)

or \( \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \)
Example

Diagonalize if possible.

\[
\begin{pmatrix}
3 & 1 \\
0 & 3
\end{pmatrix}
\]

Eigenvalues: \( \lambda = 3 \) (alg mult = 2)

Eigenspace: \( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \) → 1-dim eigenspace.

Need two indep. eigenvectors

FAIL
More Examples

Diagonalize if possible.

\[
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 0 & 0 \\
1 & 2 & 1 \\
-1 & 0 & 1
\end{pmatrix}
\]

Same:
1. Find eigenvalues
2. Find basis for each eigenspace

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]
Distinct Eigenvalues

Fact. If $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.

Why?

$$
\begin{pmatrix}
  v_1 & v_2 & \cdots & v_n
\end{pmatrix}
\begin{pmatrix}
  \lambda_1 \\
  \vdots \\
  \lambda_n
\end{pmatrix}
\begin{pmatrix}
  v_1 \\
  \vdots \\
  v_n
\end{pmatrix}
$$

Only thing can go wrong: alg mult $> 1$.

=$(\lambda-3)^2 \lambda$
Non-Distinct Eigenvalues

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- $a_i = \text{algebraic multiplicity of } \lambda_i$
- $d_i = \text{dimension of } \lambda_i \text{ eigenspace ("geometric multiplicity")}$

Then

1. $d_i \leq a_i$ for all $i$
2. $A$ is diagonalizable $\iff \sum d_i = n$
   $\iff \sum a_i = n$ and $d_i = a_i$ for all $i$

So: if you find one eigenvalue where the geometric multiplicity is less than the algebraic multiplicity, the matrix is not diagonalizable.
Section 5.5
Complex Eigenvalues
Eigenvalues for rotations?

If \( v \) is an eigenvector of \( A \) then that means \( v \) and \( Av \) are scalar multiples, i.e. they lie on a line.

What are the eigenvectors and eigenvalues for rotation of \( \mathbb{R}^2 \) by \( \pi/2 \) (counterclockwise)?

\[
A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

\[
\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1
\]

\[
\frac{-0 \pm \sqrt{0^2 - 4}}{2} = \pm \sqrt{-1} = \pm i
\]
Imaginary numbers

Problem. When solving polynomial equations, we often run up against the issue that we can’t take the square root of a negative number:

\[ x^2 + 1 = 0 \]

Solution. Take square roots of negative numbers:

\[ x = \pm \sqrt{-1} \]

We usually write \( \sqrt{-1} \) as \( i \) (for “imaginary”), so \( x = \pm i \).

Now try solving these:

\[ x^2 + 3 = 0 \]

\[ x^2 - x + 1 = 0 \]

\[ \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}i}{2} \]
Complex numbers

The complex numbers are the numbers

\[ \mathbb{C} = \{ a + bi \mid a, b \text{ in } \mathbb{R} \} \]

We can conjugate complex numbers: \( \overline{a + bi} = a - bi \)
Complex numbers and polynomials

**Fundamental theorem of algebra.** Every polynomial of degree \( n \) has exactly \( n \) complex roots (counted with multiplicity).

**Fact.** If \( z \) is a root of a real polynomial then \( \overline{z} \) is also a root.

So what are the possibilities for degree 2, 3 polynomials?

So if \( 7+3i \) is an eigenvalue of a real matrix, \( 7-3i \) is also
Complex eigenvalues

Say $A$ is a square matrix with real entries.

We can now find complex eigenvectors and eigenvalues.

Fact. If $\lambda$ is an eigenvalue of $A$ with eigenvector $v$ then $\overline{\lambda}$ is an eigenvalue of $A$ with eigenvector $\overline{v}$.

Why?
Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

See above \( \lambda = \pm i \)

- i-eigenspace: \( \begin{pmatrix}
-i & -1 \\
1 & -i
\end{pmatrix} \)

RREF \( \begin{pmatrix}
-i & 1 \\
0 & 0
\end{pmatrix} \) \( \sim \) \( \begin{pmatrix}
-i \\
+i
\end{pmatrix} \)

- i eigenvector: \( \begin{pmatrix}
-i \\
-i
\end{pmatrix} \)

eigenspace

conjugate
Three shortcuts for complex eigenvectors

Suppose we have a $2 \times 2$ matrix with complex eigenvalue $\lambda$.

(1) We do not need to row reduce $A - \lambda I$ by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

the eigenvector is

$$A = \begin{pmatrix} -y \\ x \end{pmatrix}$$

(3) Also, we get the other eigenvalue/eigenvector pair for free: conjugation.
Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

\[
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\quad 
\begin{pmatrix}
1 & -2 \\
1 & 3
\end{pmatrix}
\quad 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -2 \\
0 & 2 & 0
\end{pmatrix}
\]