Announcements April 8

- Midterm 3 on April 17
- WeBWorK 5.5 & 5.6 due Thu Apr 9.
- Quiz 8am Fri - 8am Sat on 5.2 & 5.4.
  Expect less-Googlable questions.
- Survey about on-line learning on Canvas...
- My office hours Monday 3-4, Wed 2-3, and by appointment
- TA office hours on Blue Jeans (you can go to any of these!)
  - Isabella Wed 11-12
  - Kyle Wed 3-5, Thu 1-3
  - Kalen Mon/Wed 1-2
  - Sidhanth Tue 10-12
- Supplemental problems & practice exams on master web site
- Counseling Center: http://counseling.gatech.edu Click

Eigenvalues in Structural Engineering

Watch this video about the Tacoma Narrows bridge. ▸ Watch

Here are some toy models. ▸ Check it out

The masses move the most at their natural frequencies ω. To find those, use the spring equation: $mx'' = -kx \quad \rightsquigarrow \quad \sin(\omega t)$.

With 3 springs and 2 equal masses, we get:

$$\begin{aligned} mx_1'' &= -kx_1 + k(x_2 - x_1) \\ mx_2'' &= -kx_2 + k(x_1 - x_2) \end{aligned}$$

Guess a solution $x_1(t) = A_1(\cos(\omega t) + i\sin(\omega t))$ and similar for x_2. Finding ω reduces to finding eigenvalues of $\left(\begin{smallmatrix} -2k & k \\ k & -2k \end{smallmatrix}\right)$.

Eigenvectors: $(1, 1)$ & $(1, -1)$ (in/out of phase) ▸ Details

Section 5.5

Complex Eigenvalues

A matrix without an eigenvector

Recall that rotation matrices like

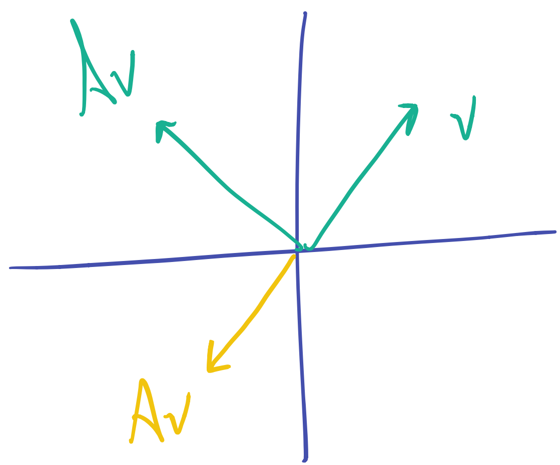
$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

have no eigenvectors. Why?

What about rotation by π? $\boxed{\lambda = -1}$



rotations:

no real eigenvectors (except 0° & 180°)

Complex numbers

The complex numbers are the numbers

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

7+3i   i

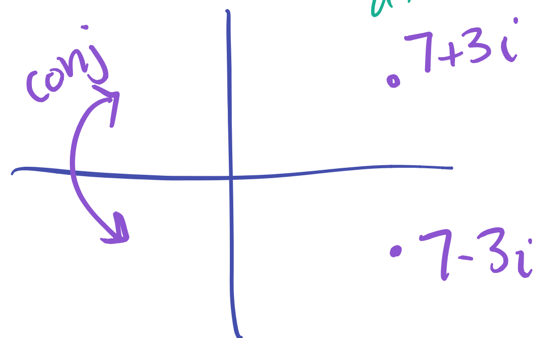
3-i   5

can add/multiply/divide.

We can conjugate complex numbers: $\overline{a + bi} = a - bi$

$$\overline{7-3i} = 7+3i$$

$$\overline{7+3i} = 7-3i$$



Fund Thm of Algebra: a polynomial of degree n has n complex roots (with multiplicity).

Complex numbers and polynomials

Eigenvalues = roots of char polys

Fundamental theorem of algebra. Every polynomial of degree n has exactly n complex roots (counted with multiplicity).

Fact. If z is a root of a real polynomial then $\overline{z}$ is also a root.

$p(z) = 0$

$\Longrightarrow p(\bar{z}) = \overline{p(z)} = 0.$

So what are the possibilities for degree 2, 3 polynomials?

So if $\underline{7-3i}$ is an eigenval. of a real matrix, $7+3i$ is also.

So complex roots come in pairs.

degree 2: can have 0 or 2 real roots.

degree 3: can have 1 or 3 real roots.

example

7, 2+i, 2-i

or 7, 1, -1

Complex eigenvalues

Say A is a square matrix with real entries.

We can now find complex eigenvectors and eigenvalues.

Fact. If λ is an eigenvalue of A with eigenvector v then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{v}$.

eigenvalue

Why?

So if $v = \begin{pmatrix} 3+i \\ i \end{pmatrix}$ is a $(5+2i)$-eigenvec

then $\underbrace{\overline{v} = \begin{pmatrix} 3-i \\ -i \end{pmatrix} \text{ is a } (5-2i)\text{-eigenval}}_{}$

conjugate everything in first line.

Three shortcuts for complex eigenvectors

Suppose we have a 2×2 matrix with complex eigenvalue λ.

(1) We do not need to row reduce $A - \lambda I$ by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

$$Av = \begin{pmatrix} xy-yx \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

the eigenvector is

$$v \; \cancel{A} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

(3) Also, we get the other eigenvalue/eigenvector pair for free: conjugation.

Complex eigenvalues

Tr(c)   Nul $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \boxed{\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

no 3x3 on exams with complex eigenvalues

Eigenvals: $\det\begin{pmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 5$

$$\rightsquigarrow \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm \sqrt{4}\sqrt{-1}}{2} = 2 \pm i$$

Eigenvecs $\boxed{2+i}$ $\begin{pmatrix} 1-(2+i) & -2 \\ 1 & 3-(2+i) \end{pmatrix} = \begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix} \xrightarrow{\text{trick 1}} \begin{pmatrix} -1-i & -2 \\ 0 & 0 \end{pmatrix}$

switch order, put a minus

trick #2 $\rightsquigarrow \begin{pmatrix} 2 \\ -1-i \end{pmatrix}$

$\boxed{2-i}$: $\begin{pmatrix} 2 \\ -1+i \end{pmatrix}$ trick 3

$\lambda = 1 : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 2i : \ldots$

Section 5.6

Stochastic Matrices (and Google!)

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow[\text{poly}]{\text{char}} (\lambda - 1)(\lambda^2 + 4)$

don't multiply out!

$\rightsquigarrow \lambda = 1, \pm 2i$

Stochastic matrices

A stochastic matrix is a non-negative square matrix where all of the columns add up to 1.

Examples:

$$\begin{pmatrix} 1/4 & 3/5 \\ 3/4 & 2/5 \end{pmatrix} \quad \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix} \quad \begin{pmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 0 & 1/4 \\ 0 & 0 & 1/4 \end{pmatrix}$$

Application: Rental Cars

Say your car rental company has 3 locations. Make a matrix whose ij entry is the fraction of cars at location i that end up at location j. For example,

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & & & \\ B & & & \\ C & & & \end{array}
\qquad
\begin{array}{c} \\ A \\ B \\ C \end{array}
\begin{array}{c} A \quad\; B \quad\; C \\ \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix} \end{array}$$

Note the columns sum to 1. Why?

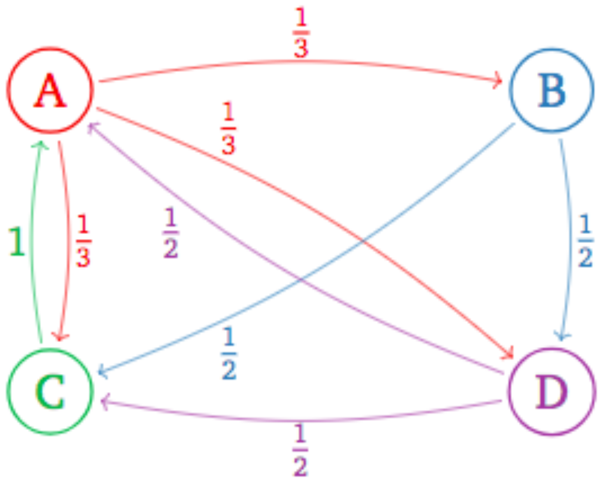
Start with 10 cars at each location.

After 1 day: $\begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix} \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ 8 \end{pmatrix}$

After $\cancel{2}$ 100 days $\begin{pmatrix} \text{orig mat} \end{pmatrix} \begin{pmatrix} 12 \\ 10 \\ 8 \end{pmatrix} = \begin{pmatrix} \text{orig} \end{pmatrix}^{\cancel{2}\ 100} \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$

Application: Web pages

Make a matrix whose ij entry is the fraction of (randomly surfing) web surfers at page i that end up at page j. If page i has N links then the ij-entry is either 0 or $1/N$.



$$\begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

Properties of stochastic matrices

Let A be a stochastic matrix.

Fact. One of the eigenvalues of A is 1 and all other eigenvalues have absolute value at most 1.

Now suppose A is a positive stochastic matrix. all entries > 0

Fact. The 1-eigenspace of A is 1-dimensional; it has a positive eigenvector.

The unique such eigenvector with entries adding to 1 is called the steady state vector.

Fact. Under iteration, all nonzero vectors approach the steady state vector.

Demo

The last fact tells us how to distribute rental cars, and also tells us the importance of web pages!

Application: Rental Cars

The rental car matrix is:

pos. stoch. matrix

$$\begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

By the prev. slide, we know:

① 1 is an eigenval

② 1-eigenspace is a line.

Its steady state vector is:

$$\begin{pmatrix} 7/18 \\ 6/18 \\ 5/18 \end{pmatrix} = \frac{1}{18}\begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix} \approx \begin{pmatrix} .39 \\ .33 \\ .28 \end{pmatrix}$$

1-eigenvector:

$Av = 1 \cdot v = v$

1-eigenvector

"steady state vector"

Application: Web pages

The web page matrix is:

$$\begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

Its steady state vector is approximately

$$\begin{pmatrix} .39 \\ .13 \\ .29 \\ .19 \end{pmatrix}$$

A is most important.

and so the first web page is the most important.

Review of Section 5.6

Can you make a stochastic matrix where the 1-eigenspace has dimension greater than 1?

Make your own internet and see if you can guess which web page is the most important. Check your answer using the method described in this section.

Summary of Section 5.6

- A stochastic matrix is a non-negative square matrix where all of the columns add up to 1.
- Every stochastic matrix has 1 as an eigenvalue, and all other eigenvalues have absolute value at most 1.
- A positive stochastic matrix has 1-dimensional eigenspace and has a positive eigenvector.
- For a positive stochastic matrix, a positive 1-eigenvector with entries adding to 1 is called a steady state vector.
- For a positive stochastic matrix, all nonzero vectors approach the steady state vector under iteration.
- Steady state vectors tell us the importance of web pages (for example).