Announcements Feb 4

- Midterm 2 on March 6
- WeBWorK 2.6 due Thursday
- My office hours Monday 3-4 and Wed 2-3
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel (different this week)
- Supplemental problems and practice exams on the master web site

1. Answer the following questions. No justification for your answer is required.

Is the matrix $\left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$ in reduced row echelon form?

YES NO

Is the vector $\binom{99}{97}$ a linear combination of the vectors $\binom{3}{4}$ and $\binom{5}{6}$?

YES NO

Suppose A is a 2×2 matrix and $A\binom{1}{1} = \binom{19}{7}$. Is it possible that the set of solutions to $Ax = 0$ is the line $x_1 = x_2$?

YES NO

Suppose A is a 4×5 matrix. Is it possible that $Ax = b$ is consistent for all b in $\mathbb{R}^4$?

YES NO

Suppose that v_1, v_2, and v_3 are vectors in $\mathbb{R}^5$. Must it be true that v_1, v_2, and v_3 are linearly independent?

YES NO

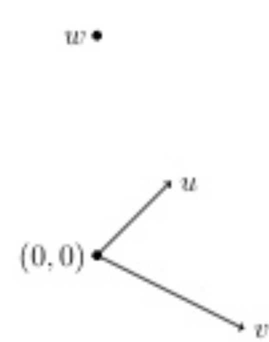
2. Answer the following questions. No justification for your answer is required.

Complete the following definition: *Vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$ are linearly independent if...*

Write down one vector in $\mathbb{R}^3$ that is not in the span of the vectors $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Find a matrix A so that the set of solutions to $Ax = 0$ is a line in $\mathbb{R}^3$ and so that the equation $Ax = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is consistent.

Circle the formula that best describes w in terms of u and v.



$u - v$ $\qquad$ $v - u$ $\qquad$ $-u - v$ $\qquad$ $2u - v$

$(\quad)\;(\,)$

5×6

3. Suppose that A is a 5×6 matrix with 2 pivots, and that $Ax = b$ is a matrix equation with b nonzero. Fill in the three blanks and answer the two multiple choice questions.

The set of solutions to $Ax = b$ is a $\boxed{4}$-dimensional plane in $\mathbb{R}^{\boxed{6}}$.

The vector b lies in $\mathbb{R}^{\boxed{5}}$.

Is the solution set to $Ax = b$ equal to a span? YES NO MAYBE

Which phrase best describes the relationship between the solutions to $Ax = 0$ and $Ax = b$?

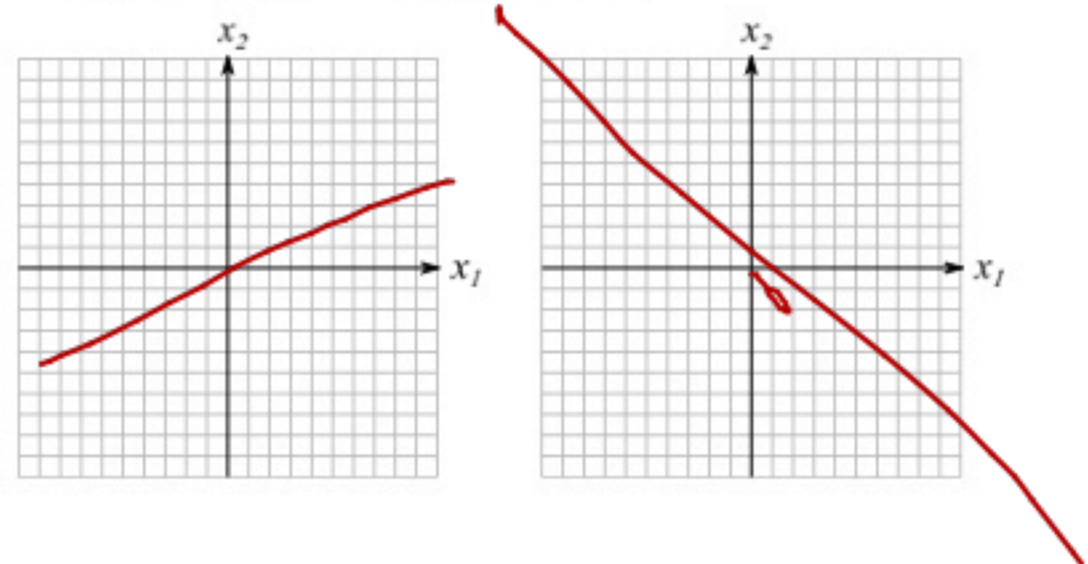
SAME PARALLEL MEET IN ONE POINT

4. Consider the matrix $A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$. Draw and label the following 5 things.

1. On the *right-hand side* draw the span of the columns of A.
2. On the *right-hand side*, draw a dot for a non-zero vector b so $Ax = b$ is consistent.
3. On the *left-hand side* draw the solutions to $Ax = b$ for your choice of b.
4. On the *left-hand side*, draw an arrow for one particular solution to $Ax = b$.
5. On the *left-hand side*, draw the solutions to $Ax = 0$.

$x_1 - 2x_2 = 0$

$\boxed{x_1 = 2x_2}$



5. Find the reduced row echelon form of the following matrix. Show your work.

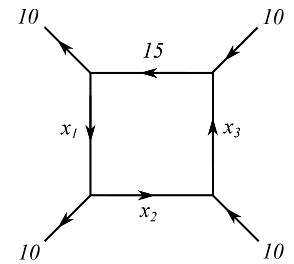
$$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 3 & -2 & 1 \\ 2 & 6 & 0 & 10 \end{pmatrix}$$

6. Suppose that there is a matrix equaion $Ax = b$ and that the reduced row echelon form of the augmented matrix $(A|b)$ is

$$\left(\begin{array}{cccc|c} 0 & 1 & -3 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Write the parametric vector form of the solution to $Ax = b$.

7. The following diagram indicates traffic flow in the town square (the numbers indicate the number of cars per minute on each section of road).



Write down a **vector equation** describing the flow of traffic. Do not solve.

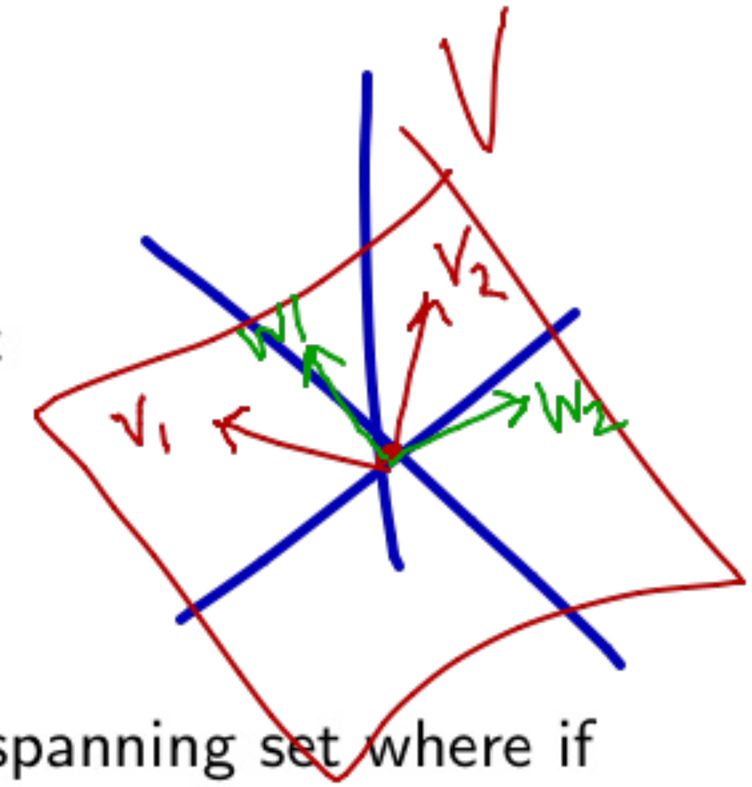
8. Find all values of h so that the vectors $\begin{pmatrix} 1 \\ 1 \\ -9 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ h \\ h \end{pmatrix}$ are linearly dependent. Show your work.

Section 2.7

Bases

Bases

V = subspace of $\mathbb{R}^n$



A basis for V is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that

1. $V = \text{Span}\{v_1, \ldots, v_k\}$
2. $v_1, \ldots, v_k$ are linearly independent

Equivalently, a basis is a *minimal spanning set*, that is, a spanning set where if you remove any one of the vectors you no longer have a spanning set.

$\dim(V)$ = dimension of $V = k =$the number of vectors in the basis

(What is the problem with this definition of dimension?)

Maybe two bases w/ diff #'s of vectors.

Q. What is one basis for $\mathbb{R}^2$? $\mathbb{R}^n$? How many bases are there?

$\hookrightarrow \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\}$

$\hookrightarrow \infty$ many It's ok.

Bases for $\mathbb{R}^n$

V

What are all bases for $\mathbb{R}^n$?

Basis for xy-plane in $\mathbb{R}^3$:
$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix}5\\7\\0\end{pmatrix}, \begin{pmatrix}9\\1\\0\end{pmatrix} \right\}$

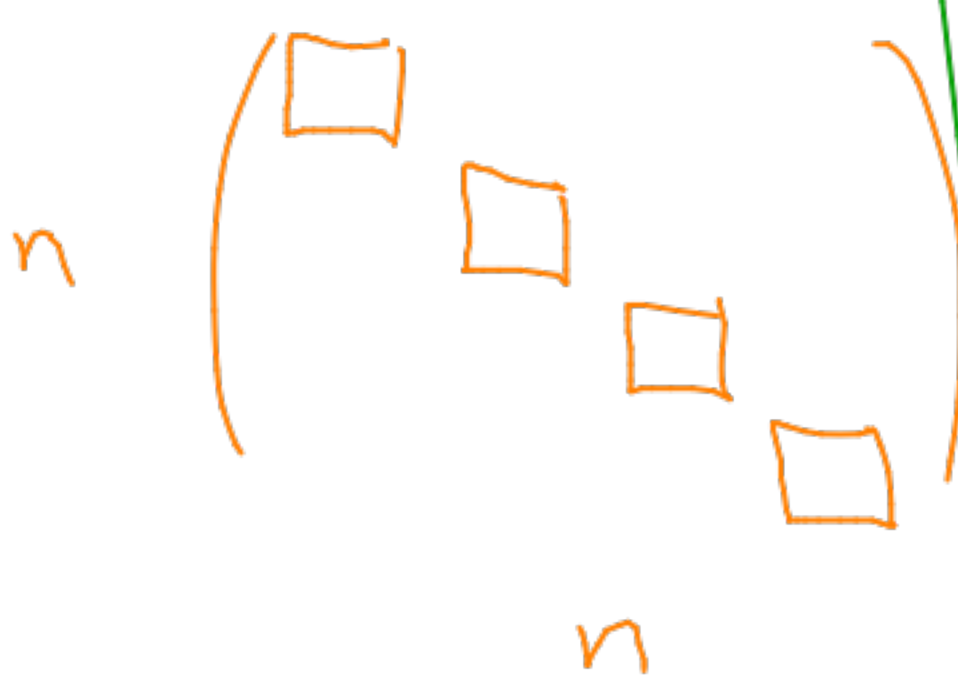
Take a set of vectors $\{v_1, \ldots, v_k\}$. Make them the columns of a matrix.

For the vectors to be linearly independent we need a pivot in every column.

For the vectors to span $\mathbb{R}^n$ we need a pivot in every row.

Conclusion: (1) $k = n$ and the matrix has n pivots.

(2) $\dim \mathbb{R}^n = n$

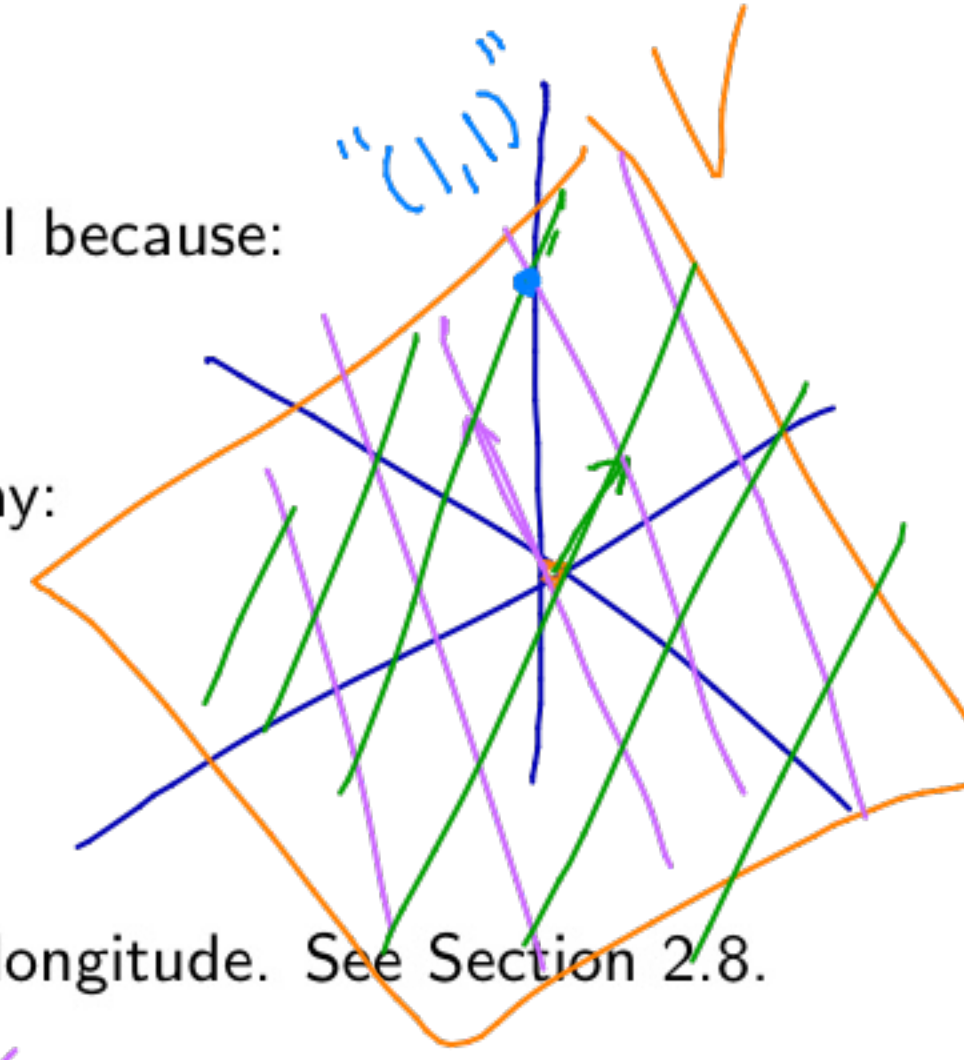


Is $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}$ a basis for $\mathbb{R}^3$?

$$\rightsquigarrow \begin{pmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{pmatrix} \xrightarrow[\text{red}]{\text{row}} \begin{pmatrix}* & * & *\\ & * & *\\ 0 & 0 & 0\end{pmatrix}$$ NO

Who cares about bases

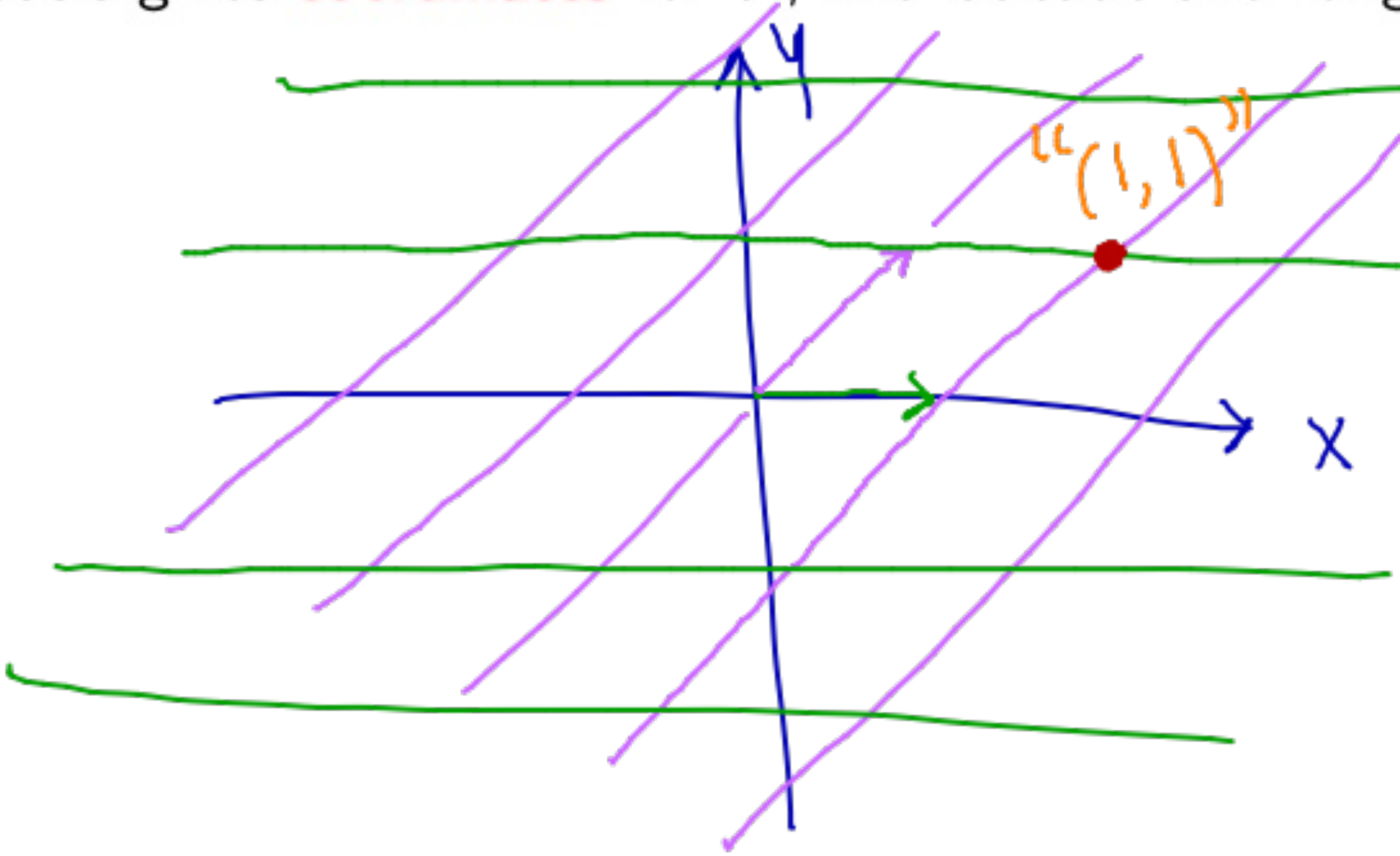
A basis $\{v_1, \ldots, v_k\}$ for a subspace V of $\mathbb{R}^n$ is useful because:



Every vector v in V can be written in exactly one way:

$$v = c_1 v_1 + \cdots + c_k v_k$$

So a basis gives coordinates for V, like latitude and longitude. See Section 2.8.



Basis:

$$\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{u}, \quad \underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{v}$$

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find vect. param form...

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Basis for $\mathrm{Nul}(A)$

$$\begin{aligned} x &= -y-z \\ y &= y \\ z &= z \end{aligned} \qquad y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Basis for $\mathrm{Col}(A)$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

line, 1-dim.

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

basis for Col(A)

$$\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}$$

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

In general:

- our usual parametric solution for $Ax = 0$ gives a basis for $\mathrm{Nul}(A)$
- the pivot columns of A form a basis for $\mathrm{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

What should you do if you are asked to find a basis for Span$\{v_1, \ldots, v_k\}$?

Find basis for $\mathrm{Col}(A)$ $\quad A = \begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix}$

Bases for planes

Find a basis for the plane $2x + 3y + z = 0$ in $\mathbb{R}^3$.

This plane is Nul(A)

$A = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$

$\rightsquigarrow$ vect. param. form

$\rightsquigarrow$ 2 vectors.

Basis theorem

Basis Theorem

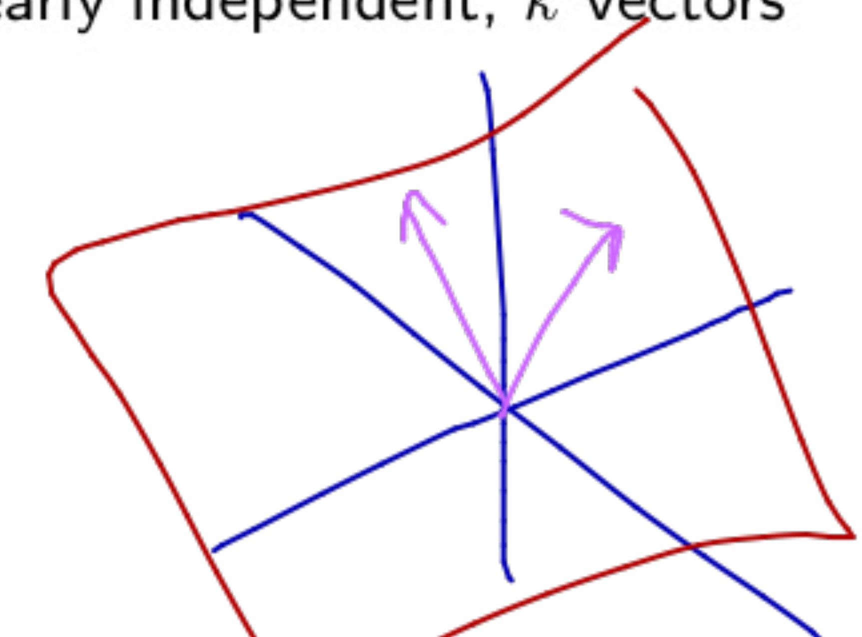
If V is a k-dimensional subspace of $\mathbb{R}^n$, then

- any k linearly independent vectors of V form a basis for V
- any k vectors that span V form a basis for V

In other words if a set has two of these three properties, it is a basis:

spans V, linearly independent, k vectors

get this for free



We are skipping Section 2.8 this semester. But remember: the whole point of a basis is that it gives coordinates (like latitude and longitude) for a subspace. Every point has a unique address.

Section 2.7 Summary

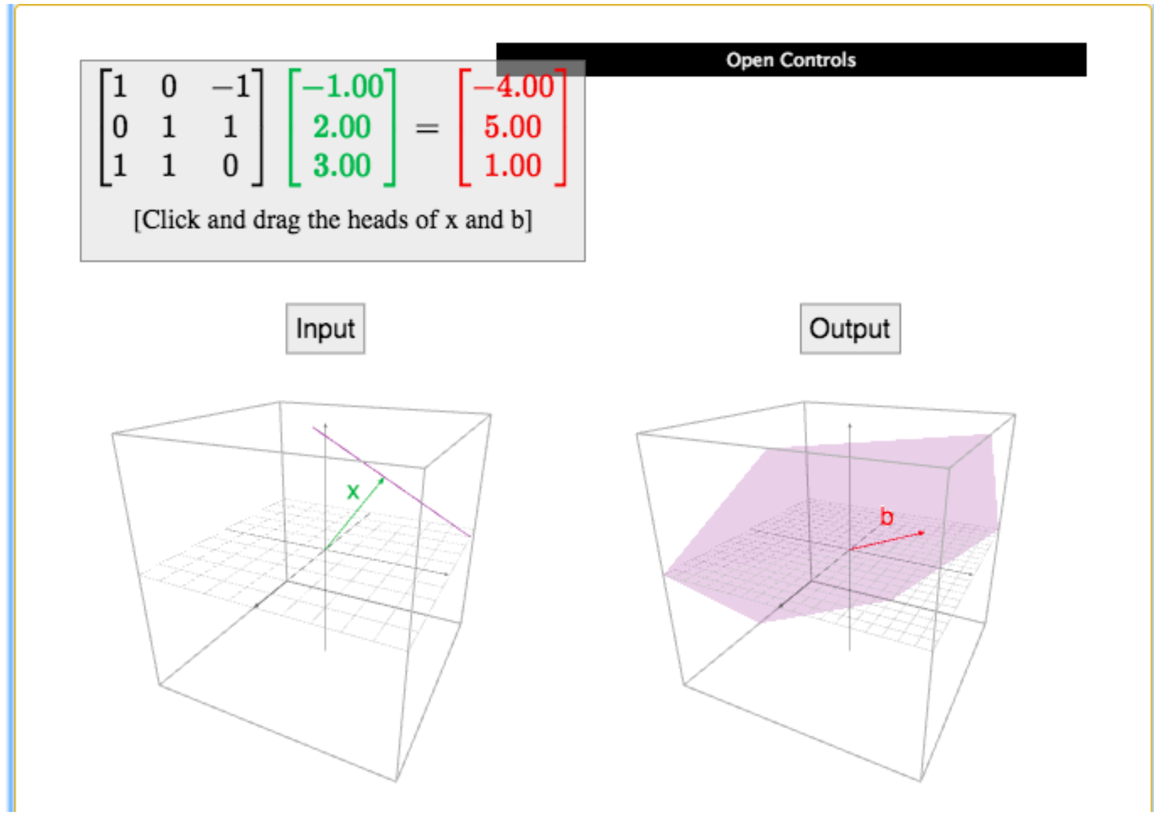
- A basis for a subspace V is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that
  1. $V = \text{Span}\{v_1, \ldots, v_k\}$
  2. $v_1, \ldots, v_k$ are linearly independent
- The number of vectors in a basis for a subspace is the dimension.
- Find a basis for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a basis for $\mathrm{Col}(A)$ by taking pivot columns of A (not reduced A)
- Basis Theorem. Suppose V is a k-dimensional subspace of $\mathbb{R}^n$. Then
  - Any k linearly independent vectors in V form a basis for V.
  - Any k vectors in V that span V form a basis.

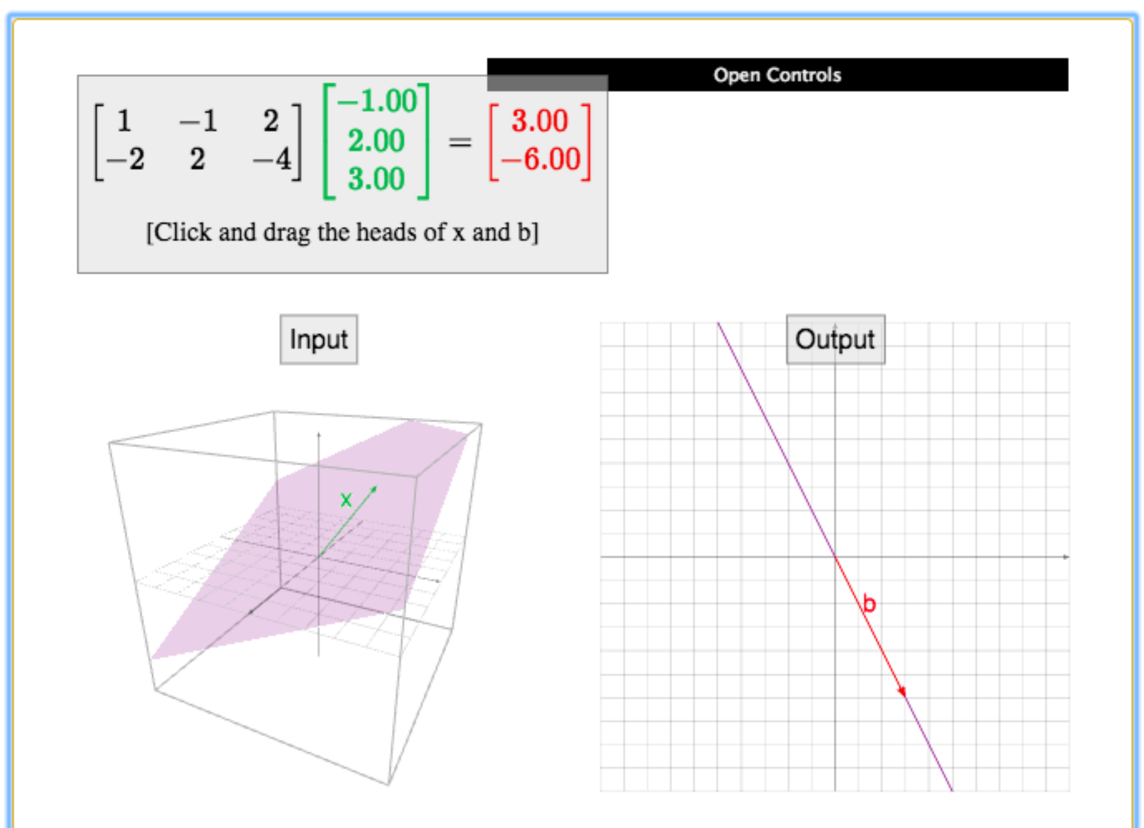
Section 2.9

The rank theorem

Rank Theorem

On the left are solutions to $Ax = 0$, on the right is $\mathrm{Col}(A)$:





Rank Theorem

$$\text{rank}(A) = \dim \text{Col}(A) = \#\ \text{pivot columns}$$
$$\text{nullity}(A) = \dim \text{Nul}(A) = \#\ \text{nonpivot columns}$$

Rank-Nullity Theorem. $\text{rank}(A) + \text{nullity}(A) = \#\text{cols}(A)$

This ties together everything in the whole chapter: rank A describes the b's so that $Ax = b$ is consistent and the nullity describes the solutions to $Ax = 0$. So more flexibility with b means less flexibility with x, and vice versa.

Example. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Section 2.9 Summary

- Rank Theorem. $\text{rank}(A) + \dim \text{Nul}(A) = \#\text{cols}(A)$