Announcements Feb 12

- Midterm 2 on March 6
- WeBWorK 2.6 due Thursday
- My office hours Monday 3-4 and Wed 2-3

Skiles 234

- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel (different this week)
- Supplemental problems and practice exams on the master web site

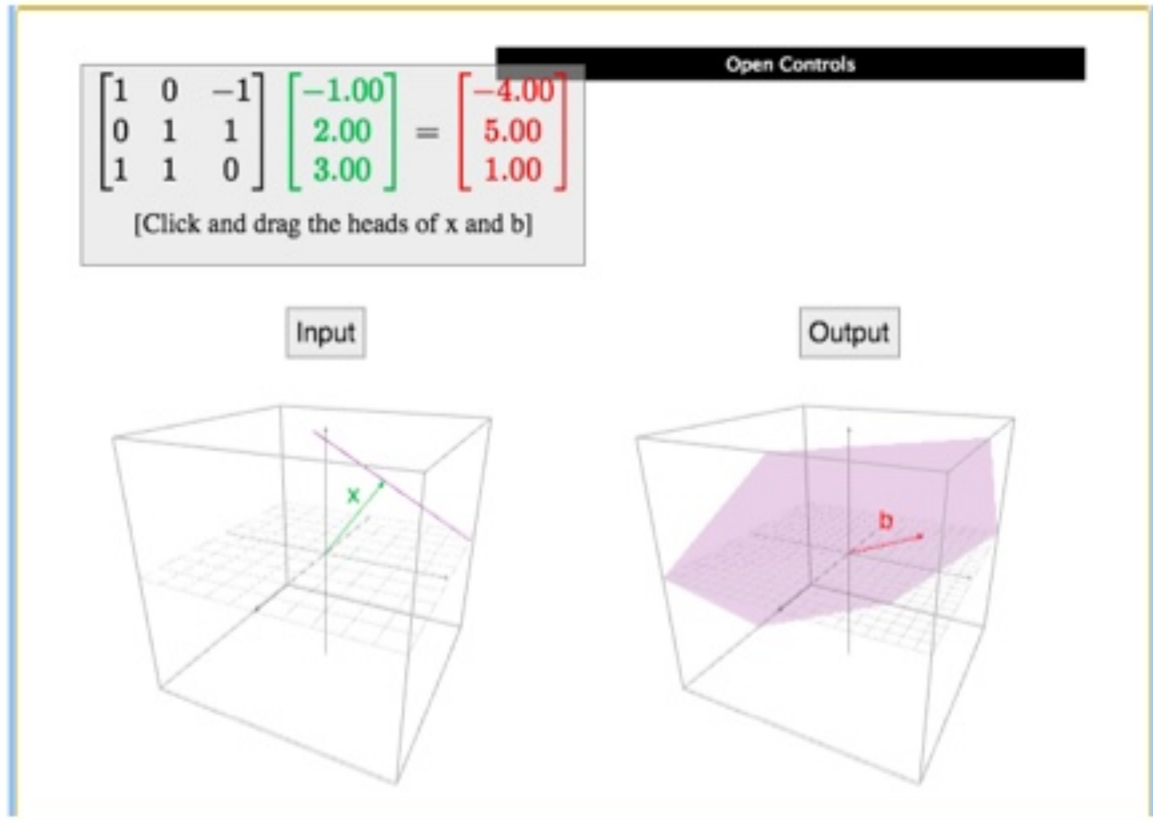
Section 2.9

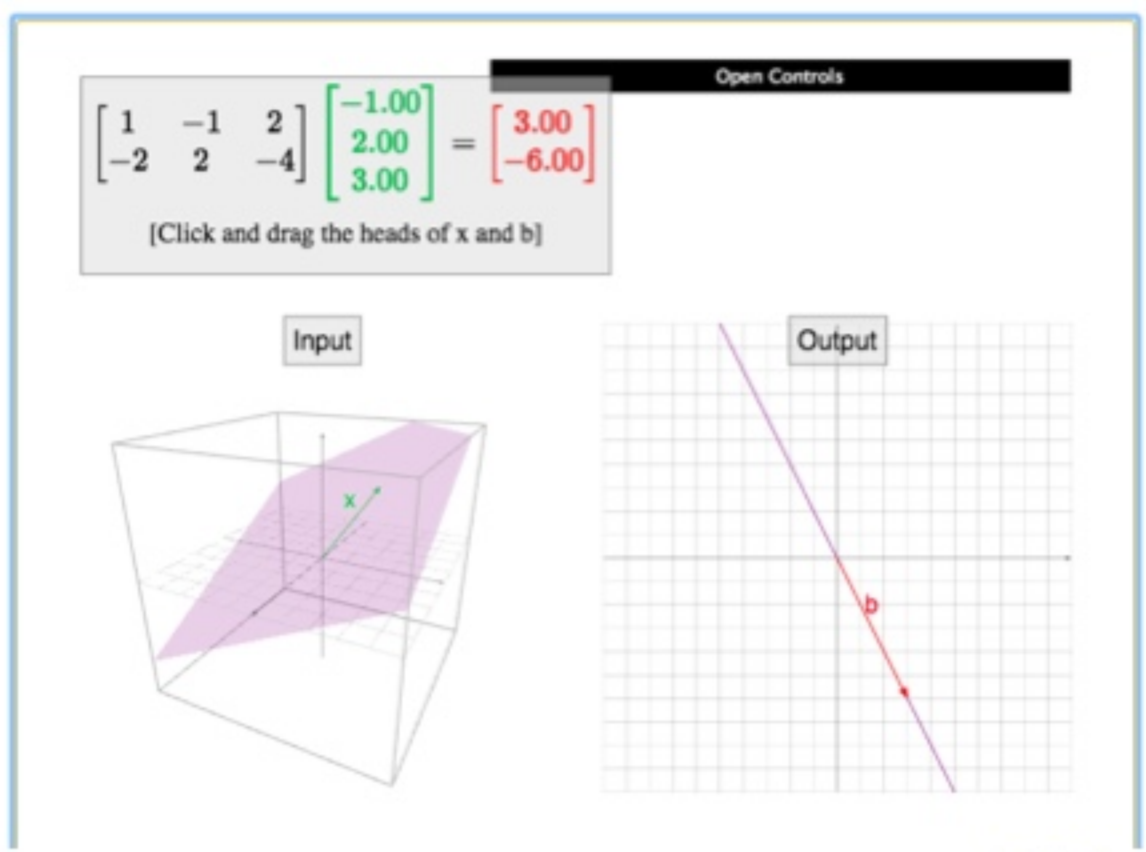
The rank theorem

Rank Theorem

On the left are solutions to $Ax = b$, on the right is $\mathrm{Col}(A)$:

all b's so that Ax=b is consistent





Rank Theorem

$$\text{rank}(A) = \dim \text{Col}(A) = \#\text{ pivot columns}$$
$$\text{nullity}(A) = \dim \text{Nul}(A) = \#\text{ nonpivot columns}$$

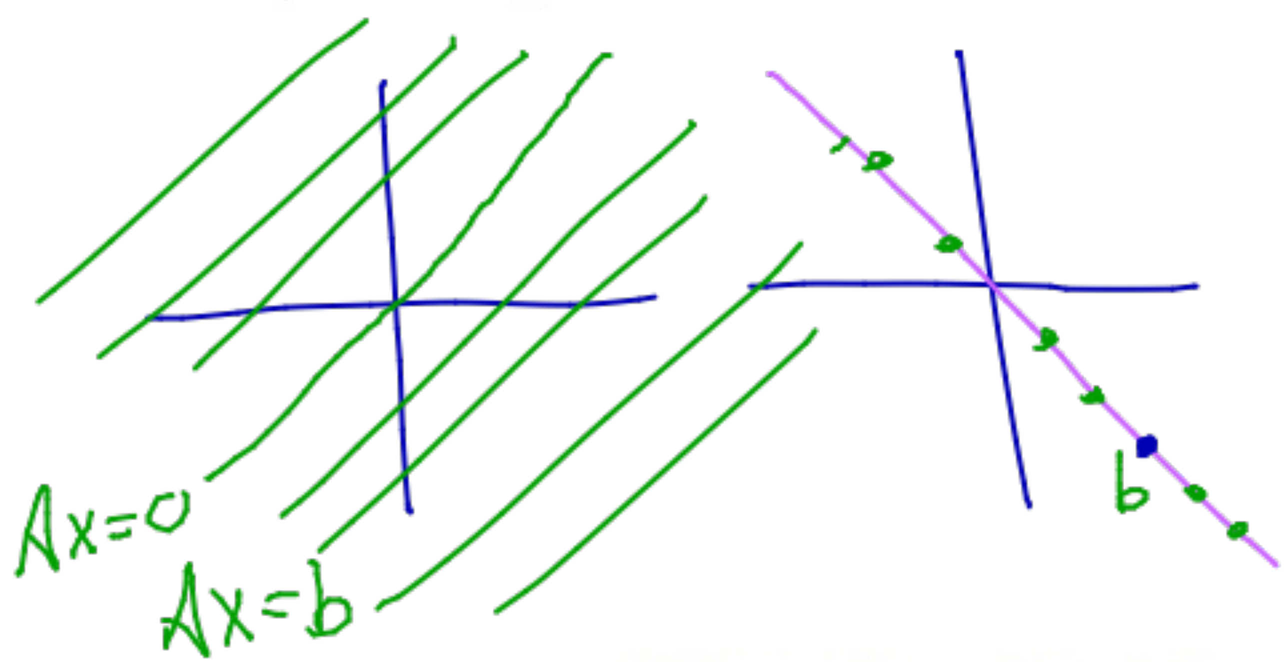
Rank-Nullity Theorem. $\text{rank}(A) + \text{nullity}(A) = \#\text{cols}(A)$

$\boxed{Ax=b}$

same dim.

This ties together everything in the whole chapter: rank A describes the b's so that $Ax = b$ is consistent and the nullity describes the solutions to $\boxed{Ax = 0.}$ So more flexibility with b means less flexibility with x, and vice versa.

Example. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$



Section 2.9 Summary

- Rank Theorem. $\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = \#\mathrm{cols}(A)$

Sections 3.1

Matrix Transformations

Section 3.1 Outline

- Learn to think of matrices as functions, called matrix transformations
- Learn the associated terminology: domain, codomain, range
- Understand what certain matrices do to $\mathbb{R}^n$

$$\underset{m \times n}{\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}} \underset{n \times 1}{\begin{pmatrix} \quad \\ \\ \\ \end{pmatrix}} = \underset{m \times 1}{\begin{pmatrix} \quad \\ \\ \\ \end{pmatrix}}$$

From matrices to functions

Let A be an $m \times n$ matrix.

We define a function

$$T : \mathbb{R}^n \to \mathbb{R}^m$$
$$T(v) = Av$$

This is called a matrix transformation.

all possible inputs

The domain of T is $\mathbb{R}^n$.

all possible outputs

The co-domain of T is $\mathbb{R}^m$.

all outputs

The range of T is the set of outputs: $\mathrm{Col}(A)$

This gives us *another* point of view of $Ax = b$

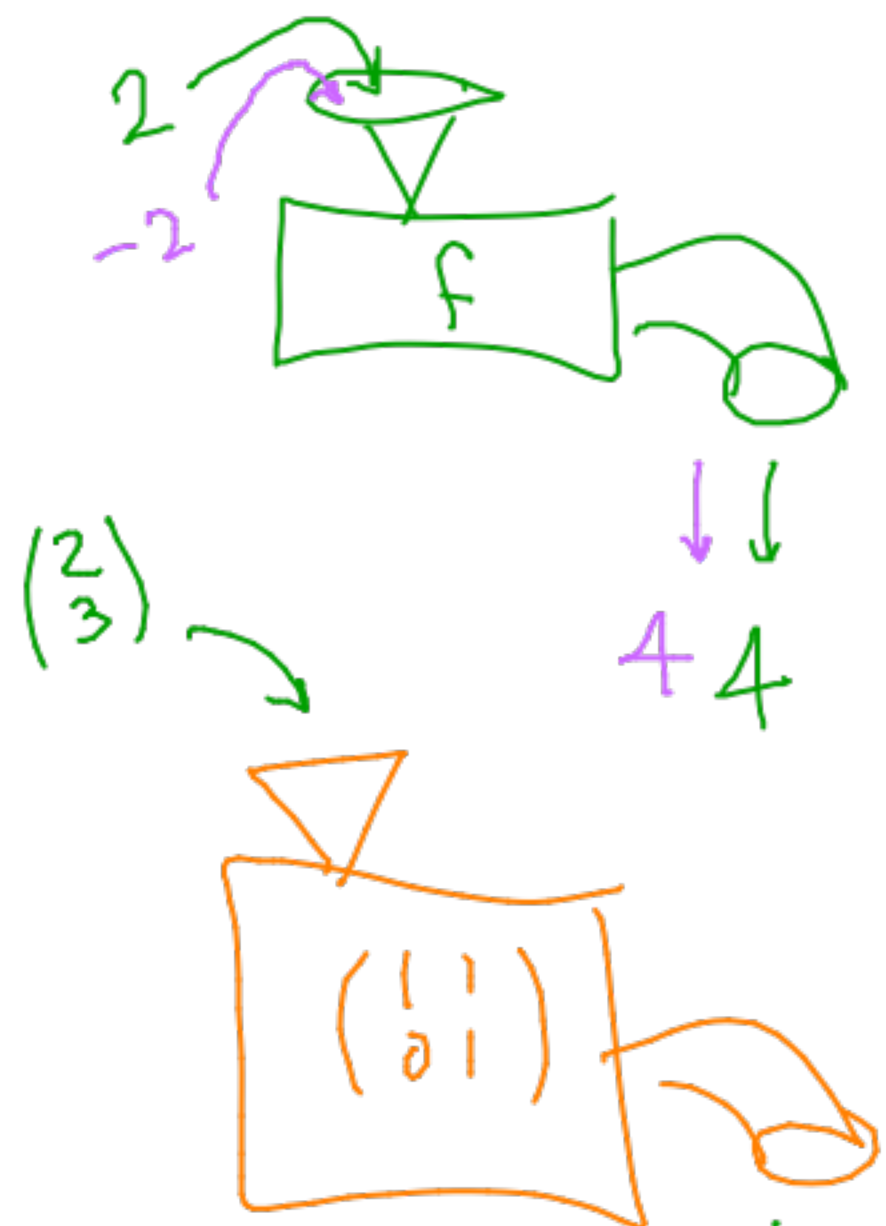
$T(x) = b.$

Demo

From Calc

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ 3 \end{pmatrix}}$$

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$.

What is $T(u)$?

$$Au = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$$

Find v in $\mathbb{R}^2$ so that $T(v) = b$

Solve $Ax = b$ $\left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array}\right)$.

$$\boxed{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}$$

Find a vector in $\mathbb{R}^3$ that is not in the range of T.

$$\begin{pmatrix} 5 \\ 5 \\ 6 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

1st & 3rd entries different

Square matrices

For a square matrix we can think of the associated matrix transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

as doing something to $\mathbb{R}^n$.

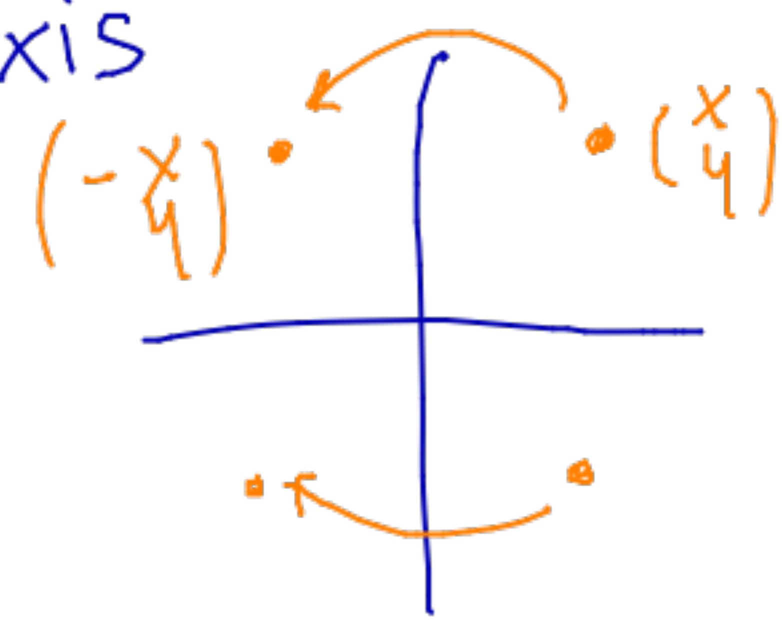
Example. The matrix transformation T for

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

What does T do to $\mathbb{R}^2$?

Reflection about y-axis

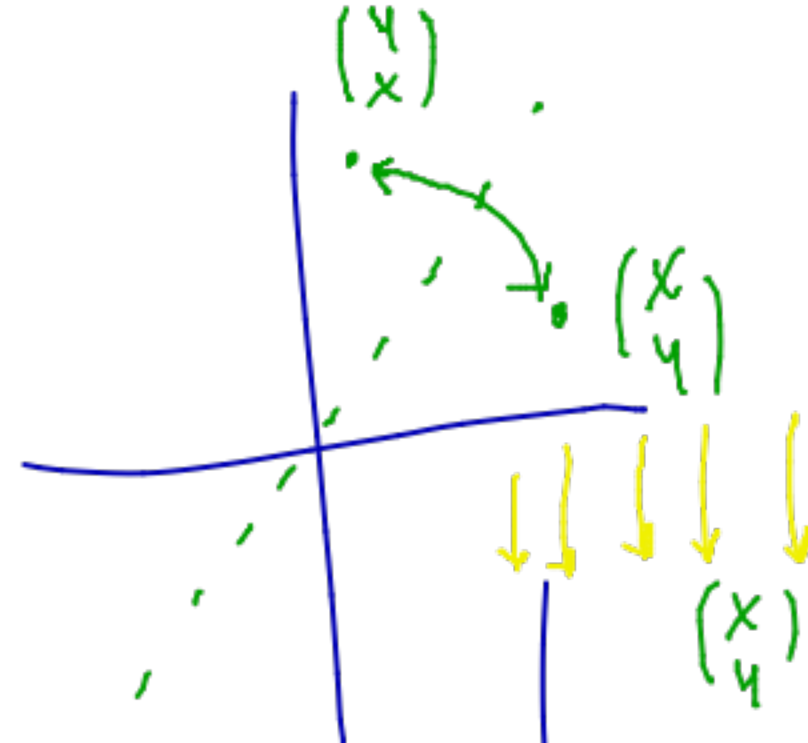


Square matrices

What does each matrix do to $\mathbb{R}^2$?

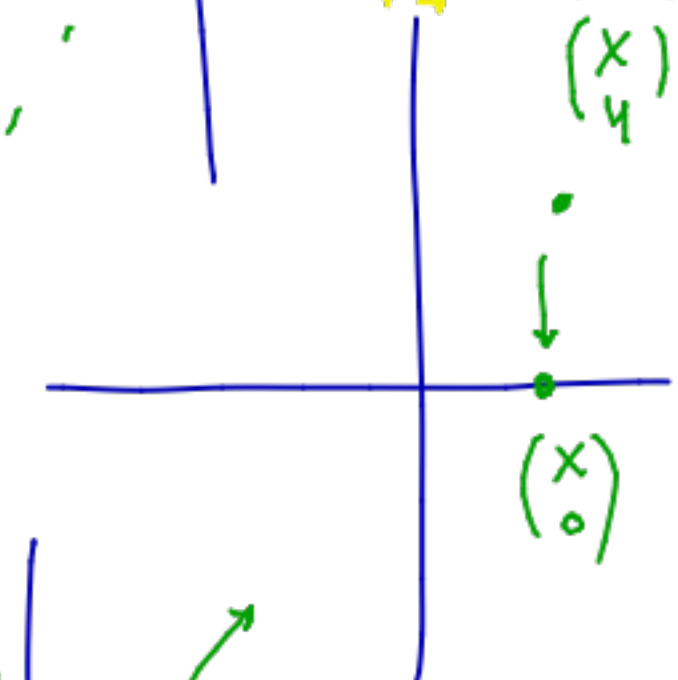
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

Reflect. about $y=x$



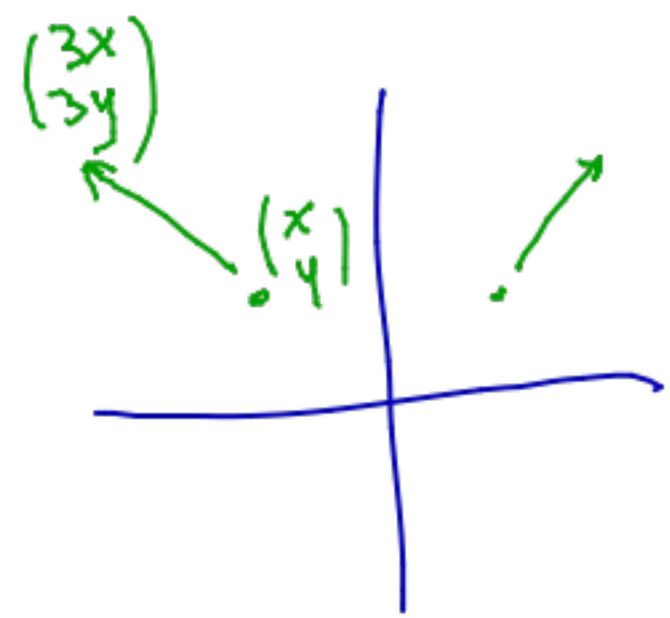
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

Projection to x-axis



$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$$

Dilate by 3



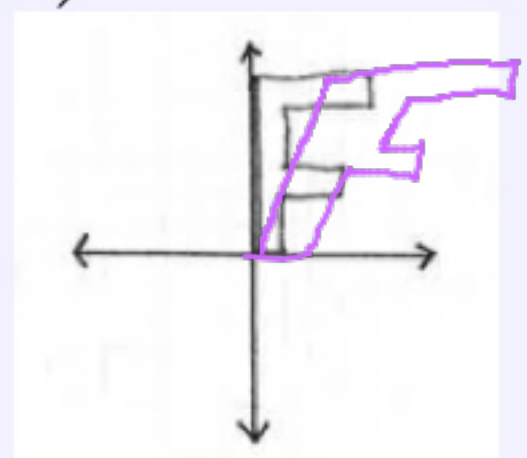
What is the range in each case?

$\mathbb{R}^2$, x-axis in $\mathbb{R}^2$, $\mathbb{R}^2$

Poll

Poll

What does $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ do to this letter F?



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$$

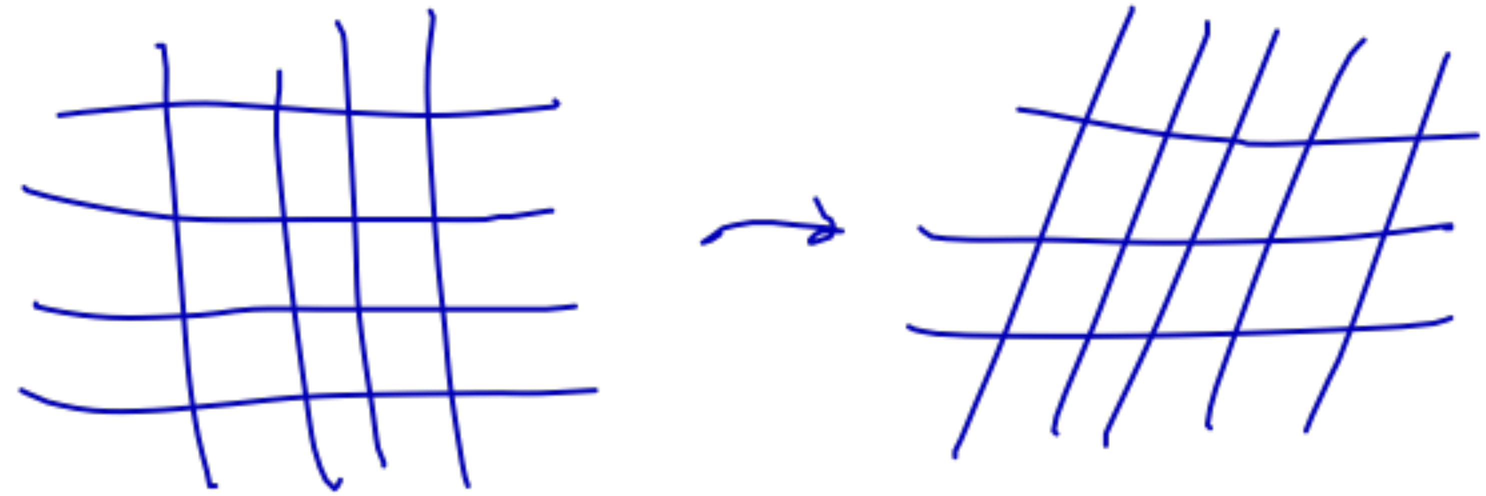
Square matrices

What does each matrix do to $\mathbb{R}^2$?

Hint: if you can't see it all at once, see what happens to the x- and y-axes.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

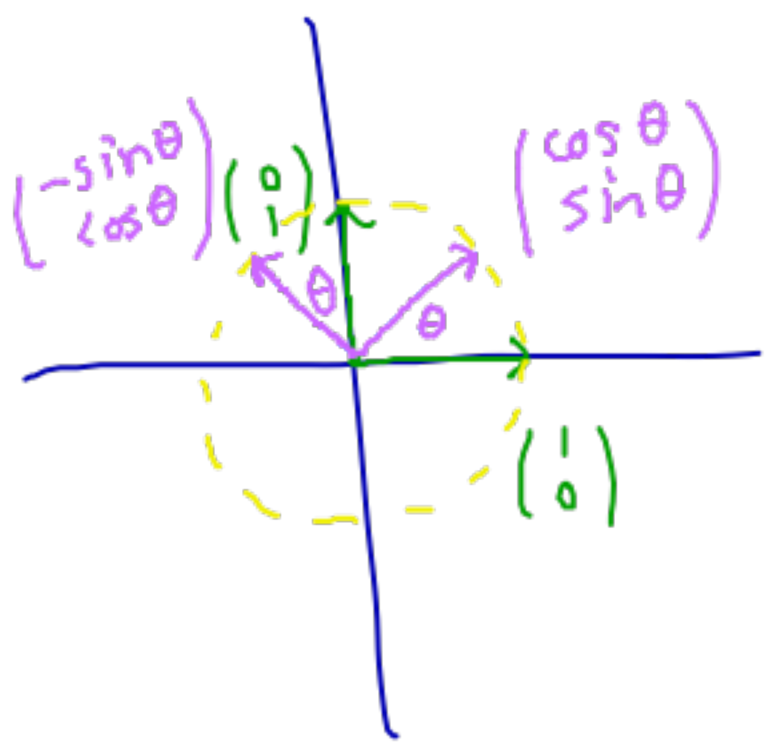
Shear.



$$\boxed{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

θ = fixed number



$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1^{st}$ col.

$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2^{nd}$ col.

Rotation by θ

Examples in $\mathbb{R}^3$

What does each matrix do to $\mathbb{R}^3$?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Projection to xy-plane

Codom: $\mathbb{R}^3$

Range: xy-plane (in $\mathbb{R}^3$)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Section 3.1 Summary

- If A is an $m \times n$ matrix, then the associated matrix transformation T is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\mathrm{Col}(A)$.
- If A is $n \times n$ then T does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation

Sections 3.2

One-to-one and onto transformations

Section 3.2 Outline

- Learn the definitions of one-to-one and onto functions
- Determine if a given matrix transformation is one-to-one and/or onto

One-to-one

$T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

In other words: different inputs have different outputs.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is one-to-one
- the columns of A are linearly independent
- $Ax = 0$ has only the trivial solution
- A has a pivot in each column
- the range of T has dimension n

What can we say about the relative sizes of m and n if T is one-to-one?

Draw a picture of the range of a one-to-one mapping $\mathbb{R} \to \mathbb{R}^3$.

Onto

$T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is onto
- the columns of A span $\mathbb{R}^m$
- A has a pivot in each row
- $Ax = b$ is consistent for all b in $\mathbb{R}^m$
- the range of T has dimension m

What can we say about the relative sizes of m and n if T is onto?

Give an example of an onto mapping $\mathbb{R}^3 \to \mathbb{R}$.

One-to-one and Onto

Do the following give matrix transformations that are one-to-one? onto?

$$\begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

One-to-one and Onto

Which of the previously-studied matrix transformations of $\mathbb{R}^2$ are one-to-one? Onto?

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ reflection

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projection

$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ scaling

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ shear

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotation

Summary of Section 3.2

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is one-to-one
  - the columns of A are linearly independent
  - $Ax = 0$ has only the trivial solution
  - A has a pivot in each column
  - the range has dimension n
- $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is onto
  - the columns of A span $\mathbb{R}^m$
  - A has a pivot in each row
  - $Ax = b$ is consistent for all b in $\mathbb{R}^m$.
  - the range of T has dimension m